COMPLEX MANIFOLDS
AND
DEFORMATION THEORY

BY

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Abstract

This thesis is a survey on the fundamental work of K.Kodaira and D.C.Spencer on the deformation of compact complex manifolds. Since a compact complex manifold M is composed of a finite number of coordinate neighbourhoods patched together, its deformation is obtained by changing the way of patching. In chapter 1, we will show that an infinitesimal deformation is represented by an element in the cohomology group $H^1(M, \Theta)$ of M with coefficients in the sheaf Θ of germs of holomorphic vector fields. Naturally, one would ask whether any element in $H^1(M, \Theta)$ represents an infinitesimal deformation. It turns out that if $H^2(M, \Theta) = 0$, then the answer is positive. This is presented in chapter 2. To consider all the infinitesimal deformations of M, we consider the number of moduli $m(M)$ of M in chapter 3. Moreover, some theorems about the relation between $m(M)$ and $\dim H^1(M, \Theta)$, and examples are included in this chapter. Finally, a completeness theorem on deformation is proved in chapter 4.

Contents

Chapter 1

Infinitesimal Deformation of Compact Complex Manifolds

1.1 Differentiable Family

We first consider a differentiable family $\{M_t\}$ of compact complex manifolds parametrized by points t in a domain B in $\mathbb{R}^m$. The precise definition is as follows.

Definition 1.1.1 *$\{M_t : t \in B\}$ is called a differentiable family of compact complex manifolds if there is a differentiable manifold $\mathcal{M}$ and a C^∞ map $\hat{\omega}$ of $\mathcal{M}$ onto B satisfying the following conditions:*

1. *The rank of the Jacobian matrix of $\hat{\omega}$ is equal to m at every point of $\mathcal{M}$.*

2. *For each $t \in B$, $\hat{\omega}^{-1}(t)$ is a compact connected subset.*

3. $M_t = \hat{\omega}^{-1}(t)$.

4. There are a locally finite open covering $\{\mathcal{U}_j : j = 1, 2, \ldots\}$ *of* $\mathcal{M}$ *and complex-valued* C^∞ *functions* $z_j^1(p), \ldots, z_j^n(p), j = 1, 2, \ldots,$ *defined on* $\mathcal{M}$ *such that for each* $t, \{p \to (z_j^1(p), \ldots, z_j^n(p)) : \mathcal{U}_j \cap \hat{\omega}^{-1}(t) \neq \emptyset\}$ *form a system of local complex coordinates of* M_t.

Notation and Terminology

We denote the differentiable family $\{M_t : t \in B\}$ by $(\mathcal{M}, B, \hat{\omega})$, and call B its parameter space and $t \in B$ its parameter. Set

$$x_j(p) = (z_j^1(p), \ldots, z_j^n(p), t_1, \ldots, t_m),\ t = (t_1, \ldots, t_m) = \hat{\omega}(p).$$

$\{x_j\}$ is called a system of local coordinates of the differentiable family.

Definition 1.1.2 *Let* M *and* N *be two compact complex manifolds.* M *is called a deformation of* N *if* M *and* N *both belong to some differentiable family.*

By standard techniques using differentiable vector fields, it can be shown that compact complex manifolds in the same differentiable family are diffeomorphic. This can be formulated locally as follows:

Theorem 1.1.3 *In Definition 1.1.1, suppose* $0 \in B$. *Then there exists* $U = \{t : |t_1| < r, \ldots, |t_m| < r, r > 0\} \subset B$ *and a diffeomorphism* Ψ *from* $M_0 \times U$ *onto* $\hat{\omega}^{-1}(U)$ *such that* $\hat{\omega} \circ \Psi = \pi$, *the projection map of* $M_0 \times U$ *onto* U, *and the restriction of* Ψ *to* $M_0 \times \{0\}$ *is the identity map.*

Now we explain briefly the idea behind Definition 1.1.1. Observe that we may choose systems of local coordinates $\{x_j\}$ as before such that $x_j(\mathcal{U}_j) = U_j \times I_j$, where U_j is a polydisc in $\mathbb{C}^n$ centred at the origin and I_j is a rectangle in $\mathbb{R}^m$. The coordinate transformation $x_k \to x_j$ on $\mathcal{U}_j \cap \mathcal{U}_k \neq \emptyset$ is given by

$$(z_j^1, \dots, z_j^n, t) = (f_{jk}^1(z_k, t), \dots, f_{jk}^n(z_k, t), t) = (f_{jk}(z_k, t), t) \tag{1.1}$$

where $z_k = (z_k^1, \dots, z_k^n)$ and $t = (t_1, \dots, t_m)$. By (4) of Definition1.1.1, the C^∞ functions

$$f_{jk}^\alpha(z_k, t) = f_{jk}^\alpha(z_k^1, \dots, z_k^n, t_1, \dots, t_m)$$

of $z_k^1, \dots, z_k^n, t_1, \dots, t_m$ are holomorphic in $z_k^1, \dots, z_k^n$, $\alpha = 1, \dots, n$. If we identify $p \in \mathcal{U}_j$ with $x_j(p)$, we may consider

$$\mathcal{M} = \bigcup_j (U_j \times I_j),$$

where $(z_j^1, \dots, z_j^n, t)$ and $(z_k^1, \dots, z_k^n, t)$ are identified as the same point if they satisfy (1.1). Then M_t is the compact complex manifold obtained by glueing polydisks U_j with $t \in I_j$ by identifying $z_k \in U_k$ with $z_j = f_{jk}(z_k, t) \in U_j$.

Since $\{U_j : j = 1, 2, \dots\}$ is locally finite and M_0 is compact, M_0 intersects only finitely many U_j's, say $U_1, \dots, U_l$. Then

$$M_0 \subset \bigcup_{j=1}^{l} U_j \times I, \text{ where } I = \bigcap_{j=1}^{l} I_j. \tag{1.2}$$

For each $t \in I$, $M_t = \cup_{j=1}^l U_j$ is a compact complex manifold obtained by glueing $U_1, \dots, U_l$ by identifying $z_k \in U_k$ with $z_j = f_{jk}(z_k, t) \in U_j$. The polydiscs $U_1, \dots, U_l$ are the same; only the way of glueing them depends on t. In this way, we may regard M_t, $t \in I$, as obtained from M_0 by changing the way $U_1, \dots, U_l$ are glued. At this point, the following remark is worth noting.

Remark: As each point of M_t, $t \in I$, belongs to one of U_j, the complex structure of a sufficiently small neighborhood of each point of M_t does not vary under deformation. Thus we are not considering the type of deformation in which every small portion is being changed .

1.2 Infinitesimal Deformation in Differentiable Family

It is natural to take the derivatives of the transition function $f_{jk}(z_k, t)$ with respect to the parameters $t_1, \ldots, t_m$ as the "derivatives of the complex structures". As usual in deformation theories, we shall call such derivatives "infinitesimal deformations". Initially, we shall consider families depending on one real parameter t. First we observe a basic relation on $M_t \cap \mathcal{U}_i \cap \mathcal{U}_j \cap \mathcal{U}_k \neq \emptyset$,

$$f_{ik}^{\alpha}(z_k, t) = f_{ij}^{\alpha}(f_{jk}^{1}(z_k, t), \ldots, f_{jk}^{n}(z_k, t), t), \alpha = 1, \ldots, n. \tag{1.3}$$

where $z_k = z_k(p), z_j = f_{jk}(z_k, t)$ and $t \in B \subset \mathbb{R}$.

Differentiating both sides of (1.3) with respect to t, we get

$$\frac{\partial f_{ik}^{\alpha}(z_k, t)}{\partial t} = \frac{\partial f_{ij}^{\alpha}(z_j, t)}{\partial t} + \sum_{\beta=1}^{n} \frac{\partial f_{ij}^{\alpha}(z_j, t)}{\partial z_j^{\beta}} \frac{\partial f_{jk}^{\beta}(z_k, t)}{\partial t}. \tag{1.4}$$

Writing$\frac{\partial z_i^{\alpha}}{\partial z_j^{\beta}} = \frac{\partial f_{ij}^{\alpha}(z_j, t)}{\partial z_j^{\beta}}$, we obtain

$$\frac{\partial f_{ik}^{\alpha}(z_k, t)}{\partial t} = \frac{\partial f_{ij}^{\alpha}(z_j, t)}{\partial t} + \sum_{\beta=1}^{n} \frac{\partial z_i^{\alpha}}{\partial z_j^{\beta}} \frac{\partial f_{jk}^{\beta}(z_k, t)}{\partial t}. \tag{1.5}$$

Then, by the chain rule, we have

$$\sum_{\alpha=1}^{n} \frac{f_{ik}^{\alpha}(z_k,t)}{\partial t}\frac{\partial}{\partial z_i^{\alpha}} = \sum_{\alpha=1}^{n} \frac{\partial f_{ij}^{\alpha}(z_j,t)}{\partial t}\frac{\partial}{\partial z_i^{\alpha}} + \sum_{\beta=1}^{n} \frac{\partial f_{jk}^{\beta}(z_k,t)}{\partial t}\frac{\partial}{\partial z_j^{\beta}}. \tag{1.6}$$

Now, we consider the vector fields $\theta_{jk}(t)$ on $\mathcal{U}_j \cap \mathcal{U}_k \neq \emptyset$, given by

$$\theta_{jk}(t) = \sum_{\alpha=1}^{n} \frac{\partial f_{jk}^{\alpha}(z_k,t)}{\partial t}\frac{\partial}{\partial z_j^{\alpha}}. \tag{1.7}$$

Observe that $\theta_{jk}(t)$ is holomorphic along each M_t. Further, (1.6) becomes a cocycle relation

$$\theta_{ik}(t) = \theta_{ij}(t) + \theta_{jk}(t). \tag{1.8}$$

Note that $f_{ii}^{\alpha}(z_i,t) = z_i^{\alpha}$. Hence $\theta_{ii}(t) = 0$. Then (1.8) gives $\theta_{kj}(t) = -\theta_{jk}(t)$. Thus we are led to a cocycle $\{\theta_{jk}(t)\} \in Z^1(\mathcal{U}_t, \Theta_t)$ of the sheaf Θ_t of germs of holomorphic vector fields over M_t with respect to the open covering $\mathcal{U}_t = \{\mathcal{U}_{jt}\}$, where $\mathcal{U}_{jt} = \mathcal{U}_j \cap M_t \neq \emptyset$. Let $\theta(t) \in H^1(M_t, \Theta_t)$ be the cohomology class of $\{\theta_{jk}(t)\}$.

Proposition 1.2.1 *$\theta(t)$ is independent of the choice of local coordinates.*

Proof : Observe that $\theta(t)$ does not change under the refinement of the open covering $\mathcal{U} = \{\mathcal{U}_j\}$ of $\mathcal{M}$. Thus, it suffices to show that given any two local coordinates $x_j = (z_j, t)$ and $u_j = (w_j, t)$ on each $\mathcal{U}_j$, $\theta(t)$ defined with respect to $\{x_j\}$ coincides with $\eta(t)$ defined with respect to $\{u_j\}$. Let $(w_k, t) \to (w_j, t)$=$(h_{jk}(w_k, t), t)$ be the coordinate transformation of $\{u_j\}$ on $\mathcal{U}_j \cap \mathcal{U}_k \neq \emptyset$. Put $\eta_{jk}(t) = \sum_{\alpha=1}^{n} \frac{\partial h_{jk}^{\alpha}(w_k,t)}{\partial t}\frac{\partial}{\partial w_j^{\alpha}}$. Then $\eta(t)$ is the cohomology class of the 1-cocycle $\{\eta_{jk}(t)\} \in Z^1(\mathcal{U}_t, \Theta_t)$. Put

$$w_j^{\alpha} = g_j^{\alpha}(z_j^1, \ldots, z_j^n, t),$$

where $w_j^\alpha = w_j^\alpha(p)$, $z_j^\alpha = z_j^\alpha(p)$, $p \in \mathcal{U}_{jt}$. Then g_j^α is a C^∞ function of $z_j^1, \ldots, z_j^n, t$, which is holomorphic in $z_j^1, \ldots, z_j^n$. Note that on $\mathcal{U}_j \cap \mathcal{U}_k \neq \emptyset$, we have

$$g_j^\alpha(f_{jk}(z_k, t), t) = h_{jk}^\alpha(g_k(z_k, t), t). \tag{1.9}$$

Differentiating (1.9) in t and then considering it as vector equality as before, we get

$$\theta_{jk}(t) + \theta_j(t) = \theta_k(t) + \eta_{jk}(t). \tag{1.10}$$

where $\theta_j(t) = \sum_{\alpha=1}^{n} \frac{\partial g_j^\alpha}{\partial t} \frac{\partial}{\partial w_j^\alpha}$ is a holomorphic vector field on $\mathcal{U}_{jt}$. (1.10) implies that $\theta(t) = \eta(t)$ in $H^1(M_t, \Theta_t)$. □

Therefore, by differentiating the transition function which defines the complex structure of M_t, we finally get a cohomology class $\theta(t) \in H^1(M_t, \Theta_t)$ which can be regarded as the derivative of the complex structure of M_t.

Definition 1.2.2 *$\theta(t)$ is called the infinitesimal deformation of $M_t = \hat{\omega}^{-1}(t)$.*

As the "derivative of the complex structure of M_t with respect to t", we would like $\theta(t)$ to have the property that $\theta(t) \equiv 0$ if and only if the complex structure of M_t does not vary with t locally. For this we give the precise definition for M_t not to vary with t in the next section.

1.3 Trivial Differentiable Family

Definition 1.3.1 *Suppose given two differentiable families $(\mathcal{M}, B, \hat{\omega})$ and $(\mathcal{N}, \mathcal{B}, \pi)$ with the same base space $B \in \mathbb{R}^m$. $\mathcal{M}$ and $\mathcal{N}$ are called equivalent* *if there is a diffeomorphism Φ of $\mathcal{M}$ onto $\mathcal{N}$ such that for each $t \in B$,Φ maps $M_t = \hat{\omega}^{-1}(t)$ biholomorphically onto $\mathcal{N}_t = \pi^{-1}(t)$.*

Definition 1.3.2 *A differentiable family $(\mathcal{M}, B, \hat{\omega})$ is called trivial if it is equivalent to $(M \times B, B, \pi)$, where π is the projection map of $M \times B$ onto B and $M = \hat{\omega}^{-1}(t^0)$, $t^0 \in B$. In this case, we say that the complex structure of $M_t = \hat{\omega}^{-1}(t)$ does not vary with t.*

Definition 1.3.3 *A differentiable family $(\mathcal{M}, B, \hat{\omega})$ is called locally trivial if for each $t \in B$, there is a subdomain I with $t \in I \subset B$ such that $(\mathcal{M}_I, I, \hat{\omega})$ is trivial, where $\mathcal{M}_I = \hat{\omega}^{-1}(I)$.*

It is easy to see that if a differentiable family $(\mathcal{M}, B, \hat{\omega})$ is locally trivial, then its infinitesimal deformation $\theta(t) = 0$. In fact, since $\theta(t)$ is independent of choice of local coordinates, we may use the product coordinates (w_λ, t) on $M \times I$. Then the coordinate transformation has the form

$$(w_\lambda, t) = (h_{\lambda\mu}(w_\mu), t).$$

Hence

$$\theta_{\lambda\mu}(t) = \sum_\alpha \frac{\partial h^\alpha_{\lambda\mu}(w_\mu)}{\partial t} \frac{\partial}{\partial w^\alpha_\lambda} = 0.$$

Kodaira-Spencer [8] show that the converse is true, under a certain additional condition, by using a lemma from their theory of variation of almost complex structures.

Theorem 1.3.4 *If $dim H^1(M_t, \Theta_t)$ is independent of t and $\theta(t) \equiv 0$, then $(\mathcal{M}, B, \hat{\omega})$ is locally trivial.*

Proof : We want to show that for any $t^0 \in B$, there is an open interval $I \subset B$ containing t^0 such that $(\mathcal{M}_I, I, \hat{\omega})$ is equivalent to $(M \times I, I, \pi)$ with $M = \hat{\omega}^{-1}(t^0)$. $\theta(t) = 0$ means that there is a 0-cochain $\{\theta_j(t)\} \in C^0(\mathcal{U}_t, \Theta_t)$ such that $\{\theta_{jk}(t)\} = \delta\{\theta_j(t)\}$, that is

$$\theta_{jk}(t) = \theta_k(t) - \theta_j(t), \tag{1.11}$$

where $\theta_j(t) = \sum_{\alpha=1}^{n} \theta_j^\alpha(z_j, t)\frac{\partial}{\partial z_j^\alpha}$ and $\theta_j^\alpha(z_j, t)$ are holomorphic in z_j. However, we do not know if they are smooth in t. At this point we invoke a lemma of Kodaira- Spencer[8] which says that under the condition on $\dim H^1(M_t, \Theta_t)$, we can choose $\{\theta_j(t)\}$ such that $\theta_j^\alpha(z_j, t)$ are C^∞ in $z_j^1, \ldots z_j^n, t$. This is the only place where the $\dim H^1(M_t, \Theta_t)$ condition is needed. Note that

$$\sum_{\alpha=1}^{n} \frac{\partial f_{jk}^\alpha(z_k, t)}{\partial t}\frac{\partial}{\partial z_j^\alpha} = \sum_{\alpha=1}^{n} \theta_k^\alpha(z_k, t)\frac{\partial}{\partial z_k^\alpha} - \sum_{\alpha=1}^{n} \theta_j^\alpha(z_j, t)\frac{\partial}{\partial z_j^\alpha}.$$

Since

$$\sum_{\alpha=1}^{n} \frac{\partial f_{jk}^\alpha(z_k, t)}{\partial t}\frac{\partial}{\partial z_j^\alpha} = \left(\frac{\partial}{\partial t}\right)_k - \left(\frac{\partial}{\partial t}\right)_j, \tag{1.12}$$

we have

$$-\sum_{\alpha=1}^{n} \theta_j^\alpha(z_j, t)\frac{\partial}{\partial z_j^\alpha} + \left(\frac{\partial}{\partial t}\right)_j = -\sum_{\alpha=1}^{n} \theta_k^\alpha(z_k, t)\frac{\partial}{\partial z_k^\alpha} + \left(\frac{\partial}{\partial t}\right)_k.$$

where $\left(\frac{\partial}{\partial t}\right)_j$ denote the vector field $\frac{\partial}{\partial t}$ on $U_j \times I \subset \mathcal{M}_I$. Therefore, we may define a C^∞ vector field on $\mathcal{M}_I$ by putting on each $U_j \times I$,

$$v = -\sum_{\alpha=1}^{n} \theta_j^\alpha(z_j, t)\frac{\partial}{\partial z_j^\alpha} + \left(\frac{\partial}{\partial t}\right)_j.$$

Now, we consider the simultaneous ordinary differential equations for v :

$$\begin{cases} \frac{dz_j^\alpha}{ds} &= -\theta_j^\alpha(z_j^1, \ldots, z_j^n, t), \quad \alpha = 1, \ldots, n. \\ \frac{dt}{ds} &= 1 \end{cases} \tag{1.13}$$

Take any point $p \in M = \hat{\omega}^{-1}(t^0)$ and let $(\zeta_i(p), t^0)$ be the local coordinate of p, where we assume that $p \in U_i \times t^0$. Then (1.13) has the unique solution :

$$\begin{cases} z_j^\alpha(s) &= z_j^\alpha(p, s) \\ t(s) &= s \end{cases}$$

under the initial conditions:

$$\begin{cases} z_i^\alpha(t^0) &= \zeta_i^\alpha(p), \quad \alpha = 1, \ldots, n \\ t(t^0) &= t^0 \end{cases}$$

This solution gives a smooth curve on $\mathcal{M}_I$ passing through p :

$$\gamma_p : t \to \gamma_p(t) = (z_j(p,t), t), \qquad t \in I$$

Thus we obtain a family of C^∞ curves $\{\gamma_p : p \in M\}$ on $\mathcal{M}_I$. By the uniqueness theorem of the solution of simultaneous ordinary differential equations, for each point $(z_j, t) \in \mathcal{M}_I$, there exists just one curve which passes through it. Therefore the map

$$\Phi : (p, t) \to (z_j, t) = (z_j(p,t), t)$$

is a diffeomorphism of $M \times I$ onto $\mathcal{M}_I$. Obviously, Φ maps $M \times t$ bijectively onto M_t. Thus in order to show that $\Phi : M \times t \to M_t$ is biholomorphic, it suffices to prove that $z_j(p,t)$ is holomorphic in p. By (1.13), we have

$$\frac{d}{dt} z_j^\alpha(\zeta_i, t) = -\theta_j^\alpha(z_j^1(\zeta_i, t), \ldots, z_j^n(\zeta_i, t), t), \tag{1.14}$$

where $\zeta_i = \zeta_i(p)$. Since $\theta_j^\alpha(z_j^1, \ldots, z_j^n, t)$ is holomorphic in $z_j^1, \ldots, z_j^n$, by applying $\frac{\partial}{\partial \bar{\zeta}_i^\lambda}$ to (1.14), we get

$$\frac{d}{dt} \frac{\partial z_j^\alpha(\zeta_i, t)}{\partial \bar{\zeta}_i^\lambda} = -\sum_{\beta=1}^n \frac{\partial \theta_j^\alpha(z_j(\zeta_i, t), t)}{\partial z_j^\beta} \frac{\partial z_j^\beta(\zeta_i, t)}{\partial \bar{\zeta}_i^\beta}.$$

As $z_j^\alpha(\zeta_i, t^0) = \zeta_i$, we have $\frac{\partial z_j^\alpha(\zeta_i,t^0)}{\partial \bar{\zeta}_i^\beta} = 0$. Hence, by the uniqueness of the solution, $\frac{\partial z_j^\alpha(\zeta_i,t)}{\partial \bar{\zeta}_i^\alpha} \equiv 0$, that is, $z_j^\alpha(p,t)$ is holomorphic in p. □

To extend the notion of the infinitesimal deformation to more than one parameter, we define a map $\rho_t : T_t(B) \to H^1(M_t, \Theta_t)$ by $\rho_t(\frac{\partial}{\partial t}) = \frac{\partial M_t}{\partial t}$ for each $\frac{\partial}{\partial t} \in T_t(B)$. Note that ρ_t is an $\mathbb{R}$-linear map of $T_t(B)$ into $H^1(M_t, \Theta_t)$.

Theorem 1.3.5 *If $dim H^1(M_t, \Theta_t)$ is independent of t, then $\rho_t \equiv 0$ if and only if $(\mathcal{M}, B, \hat{\omega})$ is locally trivial.*

Proof : It is clear that the sufficient part is true. To prove the necessary part, we use induction. Consider the differentiable family $\mathcal{M}_I = \hat{\omega}^{-1}(I) = \cup_{j=1}^{l} U_j \times I$. For simplicity, we assume that $I = \{(t_1, \ldots, t_m) \ : \ |t_1| < r, \ldots, |t_m| < r, r > 0\}$. Let

$$I^{m-1} = \{(t_1, \ldots, t_{m-1}) \ : \ |t_1| < r, \ldots, |t_{m-1} < r\}$$

and

$$I_m = \{t_m \ : \ |t_m| < r\}.$$

Considering $I^{m-1} = I^{m-1} \times 0 \subset I$, we see that $(\hat{\omega}^{-1}(I^{m-1}),\ I^{m-1},\ \hat{\omega})$ is a differentiable family. As the theorem is true for $m = 1$, we may assume that $\hat{\omega}^{-1}(I^{m-1})$ is trivial. Define a map of $\hat{\omega}^{-1}(I^{m-1}) \times I_m$ onto I by $(p, t_m) \to (\hat{\omega}(p),\ t_m)$, where $p \in \hat{\omega}^{-1}(I^{m-1})$, which makes $\hat{\omega}^{-1}(I^{m-1}) \times I_m$ a differentiable family on the parameter space I. Therefore, by hypothesis, in order to prove the triviality of $\mathcal{M}_I$, it suffices to verify that $\mathcal{M}_I$ is equivalent to $\hat{\omega}^{-1}(I^{m-1}) \times I_m$. Obviously, the same argument in the proof of Theorem1.3.4 apply. □

We next quote an important upper semi-continuity theorem and refer the proof to [8]:

Theorem 1.3.6 *Given a differentiable family $(\mathcal{M},\ B,\ \hat{\omega})$, for each $s \in B$, we have $dim H^1(M_t,\ \Theta_t) \leq dim H^1(M_s,\ \Theta_s)$ if $|t-s| < \varepsilon$ provided that ε is sufficiently small.*

As a corollary of Theorem1.3.5 and Theorem1.3.6, we may recover the Frölicher-Nijenhuis theorem[3]:

Theorem 1.3.7 *Let $(\mathcal{M},\ B,\ \hat{\omega})$ be a differentiable family of compact complex manifolds, where B is a domain of $\mathbb{R}^m$ and $0 \in B$. If $H^1(M_0,\ \Theta_0) = 0$ with $M_0 = \hat{\omega}^{-1}(0)$, then for a sufficiently small open interval I with $0 \in I \subset B$, $(\mathcal{M}_I,\ I,\ \hat{\omega})$ is trivial.*

Definition 1.3.8 *A compact complex manifold M is called rigid if for any differentiable family $(\mathcal{M}, B, \hat{\omega})$ with $0 \in B$ and $\hat{\omega}^{-1}(0) = M$, there is an open interval I with $0 \in I \subset B$ such that $(\mathcal{M}_I, I, \hat{\omega})$ is trivial. Loosely speaking, the complex structure of $M = M_0$ is invariant under small perturbation of t.*

By Theorem1.3.7, if $H^1(M, \Theta) = 0$, then M is rigid.

1.4 Complex Analytic Family

We now consider a complex analytic family $\{M_t\}$ of compact complex manifolds parametrized by points in a domain B in $\mathbb{C}^m$.The precise definition is as follows:

Definition 1.4.1 *Suppose given a domain B in $\mathbb{C}^m$, and a set $\{M_t,\ t \in B\}$ of compact complex manifolds M_t depending on $t = (t_1, \ldots, t_m) \in B$. We say that M_t depends holomorphically on t and that $\{M_t : t \in B\}$ is a complex analytic family of compact complex manifolds if there is a complex manifold $\mathcal{M}$ and a holomorphic map $\hat{\omega}$ of $\mathcal{M}$ onto B satisfying the folloing conditions:*

1. *$\hat{\omega}^{-1}(t)$ is a compact complex submanifold of $\mathcal{M}$,*
2. *$M_t = \hat{\omega}^{-1}(t)$,*
3. *The rank of the Jocobian of $\hat{\omega}$ is equal to m at every point of $\mathcal{M}$.*

Following the differentiable case, we have system of local coordinates $(z_j, t) = (z_j^1, \ldots, z_j^n, t_1, \ldots, t_m)$. The transition functions :

$$z_j^\alpha = f_{jk}^\alpha(z_k^1, \ldots, z_k^n, t_1, \ldots, t_m),\ \alpha = 1, \ldots, n,$$

are then holomorphic in $z_k^1, \ldots, z_k^n, t_1, \ldots, t_m$. Further, for a tangent vector $\frac{\partial}{\partial t} \in T_t(B)$, $\theta_{jk}(t) = \sum_{\alpha=1}^{n} \frac{\partial f_{jk}^\alpha(z_k, t)}{\partial t} \frac{\partial}{\partial z_j^\alpha}$ form a cocycle of holomorphic vector fields along each M_t.

Definition 1.4.2 *The cohomology class $\theta(t) \in H^1(M_t, \Theta_t)$ of the 1-cocycle $\{\theta_{jk}(t)\}$ is called the infinitesimal deformation along $\frac{\partial}{\partial t}$ and is denoted by $\frac{\partial M_t}{\partial t}$.*

Clearly,

$$\rho_t : \frac{\partial}{\partial t} \to \rho\left(\frac{\partial}{\partial t}\right) = \frac{\partial M_t}{\partial t}$$

is a $\mathbb{C}$-linear map of $T_t(B)$ into $H^1(M_t, \Theta_t)$.

Definition 1.4.3 *Two complex analytic families* $(\mathcal{M}, B, \hat{\omega})$ *and* $(\mathcal{N}, B, \pi)$ *are called biholomorphically equivalent if there is a biholomorphic map* Φ *of* $\mathcal{M}$ *onto* $\mathcal{N}$ *such that* $\hat{\omega} = \pi \circ \Phi$.

Definition 1.4.4 *A complex analytic family* $(\mathcal{M}, B, \hat{\omega})$ *is called trivial if it is biholomorphically equivalent to* $(M \times B, B, \pi)$ *with* $M = \hat{\omega}^{-1}(t^0)$, *where* t^0 *is some point of* B.

We define the local triviality of the complex analytic family $(\mathcal{M}, B, \hat{\omega})$ in the similar way as before.

Remark(1) : As $\frac{\partial f^{\alpha}_{jk}(z_k,t)}{\partial \bar{t}} = 0$ for $\frac{\partial}{\partial \bar{t}} \in \overline{T_t(B)}$, the map ρ_t for $(\mathcal{M}, B, \hat{\omega})$ considered as a differentiable family coincides with that for $(\mathcal{M}, B, \hat{\omega})$ considered as a complex analytic family.

Theorem 1.4.5 *If* $dim H^1(M_t, \Theta_t)$ *is independent of* $t \in B$, *then* $\rho_t \equiv 0$ *if and only if* $(\mathcal{M}, B, \hat{\omega})$ *is locally trivial.*

Proof : The if part is trivial. To prove the only if part, we use induction as in the proof of Theorem1.3.5. Suppose $\mathcal{M}_\Delta = \cup_{j=1}^{l} U_j \times \Delta = \hat{\omega}^{-1}(\Delta)$, where $0 \in \Delta \subset B$ is a polydisk and each U_j is a polydisk independent of t, and $(z_j,\ t) \in U_j \times \Delta$ and $(z_k\ t) \in U_k \times \Delta$ are the same point on $\mathcal{M}_\Delta$ if $z^\alpha_j = f^\alpha_{jk}(z_k, t)$, $\alpha = 1, \ldots, n$. We assume that

$$\Delta = \{(t_1, \ldots, t_m) \ : \ |t_1| < r, \ldots, |t_m| < r,\ r > 0\}.$$

Put

$$\begin{aligned} \Delta^{m-1} &= \{(t_1, \ldots, t_m) \ : \ |t_1| < r, \ldots, |t_{m-1}| < r\}, \\ \Delta_m &= \{t_m \ : \ |t_m| < r\}, \end{aligned}$$

and consider $\Delta^{m-1} {=} \Delta^{m-1} \times 0 \subset \Delta^{m-1} \times \Delta_m {=} \Delta$. Now, in order to prove

$(\mathcal{M}, B, \hat{\omega})$ is locally trivial, it suffices to show that the complex analytic family $\hat{\omega}^{-1}(\Delta^{m-1}) \times \Delta_m$ is biholomorphically equivalent to $\mathcal{M}$. Put

$$\theta_{jk}(t) = \sum_{\alpha=1}^{n} \frac{\partial f_{jk}^{\alpha}(z_k, t_1, \ldots, t_m)}{\partial t_m} \frac{\partial}{\partial z_j^{\alpha}}.$$

Since $\rho_t \equiv 0$, there exists a 0-cochain $\{\theta_{jk}(t)\} \in C^0(\mathcal{U}_t, \Theta_t)$ such that

$$\{\theta_{jk}(t)\} = \delta\{\theta_j(t)\},$$

where $\theta_j(t) = \sum_{\alpha=1}^{n} \theta_j^{\alpha}(z_j, t) \frac{\partial}{\partial z_j^{\alpha}}$. By the method developed by Kodaira-Spencer[8], we can choose $\theta_j^{\alpha}(z_j, t)$ to be C^{∞} in $z_j^1, \ldots, z_j^n$, $t_1, \ldots, t_m$. In order to show that $\hat{\omega}^{-1}(\Delta^{m-1}) \times \Delta_m$ is biholomorphically equivalent to $\mathcal{M}_{\Delta}$, it suffices to verify that we can choose a 0-cochain $\{\theta_j(t)\}$ such that each coefficient $\theta_j^{\alpha}(z_j, t)$ of $\theta_j(t)$ is holomorphic in $z_j^1, \ldots, z_j^n$, $t_1, \ldots, t_m$. In fact if each $\theta_j^{\alpha}(z_j, t)$ is holomorphic, the map

$$\Phi \; : \; \hat{\omega}^{-1}(\Delta^{m-1}) \times \Delta_m \to \mathcal{M}_{\Delta}$$

defined by the solution of the simultaneous ordinary differential equations :

$$\begin{cases} \frac{dz_j^{\alpha}}{ds} = -\theta_j^{\alpha}(z_j, t_1, \ldots, t_m), & \alpha = 1, \ldots, n. \\ \frac{dt_{\lambda}}{ds} = 0, & \lambda = 1, \ldots, m-1. \\ \frac{dt_m}{ds} = 1 \end{cases} \tag{1.15}$$

is biholomorphic and Φ maps $\hat{\omega}^{-1}(\Delta^{m-1}) = \hat{\omega}^{-1}(\Delta^{m-1}) \times 0$ identically onto $\hat{\omega}^{-1}(\Delta^{m-1}) \subset \mathcal{M}_{\Delta}$. Let $\theta_{jk}(t) = \sum_{\alpha=1}^{n} \theta_{jk}^{\alpha}(z_j, t) \frac{\partial}{\partial z_j^{\alpha}}$, Then the equality

$$\theta_{jk}(t) = \theta_k(t) - \theta_j(t)$$

becomes

$$\theta_{jk}^{\alpha}(z_k, t) = \sum_{\beta=1}^{n} \frac{\partial z_j^{\alpha}}{\partial z_k^{\beta}} \theta_k^{\beta}(z_k, t) - \theta_j^{\alpha}(z_j, t), \; \alpha = 1, \ldots, n.$$

Since $\theta^\alpha_{jk}(z_k,t)$ and $z^\alpha_j = f^\alpha_{jk}(z_k,t)$ are holomorphic in $t_1,\dots,t_m$, we get

$$\sum_{\beta=1}^{n} \frac{\partial z^\alpha_j}{\partial z^\beta_k} \frac{\partial \theta^\beta_k(z_k,t)}{\partial \bar{t_\lambda}} = \frac{\partial \theta^\alpha_j(z_j,t)}{\partial \bar{t_\lambda}}, \ \lambda = 1,\dots,m.$$

Then, by the chain rule, we have

$$\sum_{\beta=1}^{n} \frac{\partial \theta^\beta_k(z_k,t)}{\partial \bar{t_\lambda}} \frac{\partial}{\partial z^\beta_k} = \sum_{\alpha=1}^{n} \frac{\partial \theta^\alpha_j(z_j,t)}{\partial \bar{t_\lambda}} \frac{\partial}{\partial z^\alpha_j}.$$

Thus we obtain a holomorphic vector field $\eta_\lambda(t) \in H^0(M_t,\Theta_t)$ on M_t, where $\eta_\lambda(t) = \sum\limits_{\alpha=1}^{n} \frac{\partial \theta^\alpha_j(z_j,t)}{\partial \bar{t_\lambda}} \frac{\partial}{\partial z^\alpha_j}$ on each $U_j \times t \subset M_t$.
As was mentioned before, if we consider $(\mathcal{M}, B, \hat{\omega})$ as a differentiable family, the $\mathbb{R}$-linear map ρ_t is also identically zero. Thus, by Theorem1.3.4, $(\mathcal{M}, B, \hat{\omega})$ is locally trivial, and thus $\dim H^0(M_t,\Theta_t)$ $=\dim H^0(M,\Theta)$.
Case(i) : $\dim H^0(M.\Theta) = 0$. In this case, we have $H^0(M_t,\Theta_t) = 0$. Therefore, $\eta_\lambda(t) \equiv 0$. This implies that $\theta^\alpha_j(z_j,t)$ are holomorphic in $t_1,\dots,t_m$.
Case(ii): $\dim H^0(M,\Theta) \geq 1$. In this case, we use the following lemma and refer the proof to [5].

Lemma 1.4.6 *For a complex analytic family $(\mathcal{M}, B, \hat{\omega})$ of compact complex manifolds, suppose that $\dim H^0(M_t,\Theta_t)=d$ is independent of t. Then for a sufficiently small polydisk Δ with $0 \in \Delta \subset B$, we can choose a basis $\{\varphi_1(t),\dots,\varphi_d(t)\}$ of $H^0(M_t,\Theta_t)$ with $\varphi_q(t)=\sum\limits_{\alpha=1}^{n} \varphi^\alpha_{qj}(z_j,t)\frac{\partial}{\partial z^\alpha_j}$ such that $\varphi^\alpha_{qj}(z_j,t)$ are holomorphic functions of $z^1_j,\dots,z^n_j$, $t_1,\dots,t_m$.*

Put

$$\eta_\lambda(t) = \sum_{q=1}^{d} c_{q\lambda}(t)\varphi_q(t),$$

and denote $\bar{\partial}$ with respect to $t_1, \ldots, t_m$ by $\bar{\partial}_t$. Then on each $U_j \times \Delta$, we have

$$\bar{\partial}_t \theta_j^\alpha(z_j, t) = \sum_{q=1}^{d} \varphi_{qj}^\alpha(z_j, t) \sum_{\lambda=1}^{m} c_{q\lambda}(t) d\bar{t_\lambda}.$$

Since $\bar{\partial}_t{}^2 = 0$ and $\varphi_{qj}^\alpha(z_j, t_1, \ldots, t_m)$ are holomorphic in $t_1, \ldots, t_m$, applying $\bar{\partial}_t$ to the above equality, we get

$$\sum_{q=1}^{d} \varphi_{qj}^\alpha(z_j, t) \bar{\partial}_t \left(\sum_{\lambda=1}^{m} c_{q\lambda}(t) d\bar{t_\lambda} \right) = 0.$$

This implies that

$$\bar{\partial}_t \left(\sum_{\lambda=1}^{m} c_{q\lambda}(t) d\bar{t_\lambda} \right) = 0.$$

By Dolbeault's Lemma, there exists a C^∞ function $c_q(t)$ on Δ such that

$$\bar{\partial}_t c_q(t) = \sum_{\lambda=1}^{m} c_{q\lambda}(t) d\bar{t_\lambda}.$$

for each q. Put

$$\hat{\theta}_j(t) = \theta_j(t) - \psi(t),$$

where $\psi(t) = \sum_{q=1}^{d} c_q(t) \varphi_q(t)$. Clearly, $\hat{\theta}_j(t)$ is a holomorphic vector field on $U_j \times t \subset M_t$ and

$$\theta_{jk}(t) = \hat{\theta}_k(t) - \hat{\theta}_j(t).$$

Writing $\hat{\theta}_j(t) = \sum_{\alpha=1}^{n} \hat{\theta}_j^\alpha(z_j, t) \frac{\partial}{\partial z_j^\alpha}$, we get

$$\hat{\theta}_j^\alpha(z_j, t) = \theta_j^\alpha(z_j, t) - \sum_{q=1}^{d} c_q(t) \varphi_{qj}^\alpha(z_j, t).$$

This implies that

$$\overline{\partial_t} \hat{\theta}_j^\alpha(z_j, t) = \overline{\partial_t} \theta_j^\alpha(z_j, t) - \sum_{q=1}^{d} \varphi_{qj}^\alpha(z_j, t) \overline{\partial_t} c_q(t) = 0.$$

Therefore $\hat{\theta}_j^\alpha(z_j, t)$ are holomorphic in $t_1, \ldots, t_m$. $\square$

Corollary 1.4.7 *A complex analytic family* $(\mathcal{M}, B, \hat{\omega})$ *is trivial if it is trivial as a differentiable family.*

1.5 Induced Family

Suppose given a complex analytic family $(\mathcal{M}, B, \hat{\omega})$ of compact complex manifolds, where B is a domain of $\mathbb{C}^m$. Let D be a domain of $\mathbb{C}^r$, and $h : s \to t = h(s),\ s \in D$, a holomorphic map of D into B. We define a map

$$\Pi : \mathcal{M} \times D \to B \times D \quad \text{by}$$

$$\Pi(p, s) = (\hat{\omega}(p), s).$$

Then $(\mathcal{M} \times D, B \times D, \Pi)$ is a complex analytic family with parameter space $B \times D$. Let

$$\begin{aligned} G &= \{(h(s), s) : s \in D\} \text{ and} \\ \mathcal{N} &= \Pi^{-1}(G). \end{aligned}$$

Then $(\mathcal{N}, G, \Pi)$ is a complex analytic family over the parameter space G. Since the projection

$$P : B \times D \to D$$

maps G biholomorphically onto D, we identify G with D via P. Thus we obtain the complex analytic family $(\mathcal{N}, D, \pi)$, where $\pi = P \circ \Pi$.

Definition 1.5.1 *The complex analytic family* $(\mathcal{N}, D, \pi)$ *thus obtained is called the complex analytic family induced from* $(\mathcal{M}, B, \hat{\omega})$ *by the holomorphic map* $h : D \to B$.

Since

$$\pi^{-1}(1) = \Pi^{-1}(h(s), s) = M_{h(s)} \times s,$$

we may consider

$$(\mathcal{N}, D, \pi) = \{M_{h(s)} \ : \ s \in D\}.$$

Taking a sufficiently small polydisk $E \subset D$ and let $\mathcal{N}_E = \pi^{-1}(E)$, it is clear that we may consider

$$\mathcal{N}_E = \cup_{j=1}^{l} U_j \times E \quad \text{by identifying}$$

$$(z_k, s) \in U_k \times E \quad \text{with} \quad (z_j, s) \in U_j \times E \quad \text{if} \quad z_j = f_{jk}(z_k, h(s)).$$

Theorem 1.5.2 *For any tangent vector* $\frac{\partial}{\partial s} \in T_s(D)$*, the infinitesimal deformation of* $M_{h(s)}$ *along* $\frac{\partial}{\partial s}$ *is given by*

$$\frac{\partial M_{h(s)}}{\partial s} = \sum_{\lambda=1}^{m} \frac{\partial t_\lambda}{\partial s} \frac{\partial M_t}{\partial t_\lambda}, \qquad (t_1, \ldots, t_m) = h(s).$$

Proof :

$$\begin{aligned} \sum_{\alpha=1}^{n} \frac{\partial f_{jk}^{\alpha}(z_k, h(s))}{\partial s} \frac{\partial}{\partial z_j^{\alpha}} &= \sum_{\alpha=1}^{n} \sum_{\lambda=1}^{m} \frac{\partial f_{jk}^{\alpha}(z_k, h(s))}{\partial t_\lambda} \frac{\partial t_\lambda}{\partial s} \frac{\partial}{\partial z_j^{\alpha}} \\ &= \sum_{\lambda=1}^{m} \frac{\partial t_\lambda}{\partial s} \sum_{\alpha=1}^{n} \frac{\partial f_{jk}^{\alpha}(z_k, h(s))}{\partial t_\lambda} \frac{\partial}{\partial z_j^{\alpha}} \end{aligned}$$

□

Remark(2) : If we do not assume that $\dim H^1(M_t, \Theta_t)$ is independent of t, Theorems1.3.4, 1.3.5, and 1.4.5 do not hold. In fact, let $(\mathcal{M}, \mathbb{C}, \hat{\omega})$ be the complex analytic family of Hopf surfaces M_t, each obtained as the quotient of $\mathbb{C}^2/\{0\}$ under the group of automorphisms generated by

$$g_t(z_1, z_2) = (\alpha z_1 + t z_2, \alpha z_2),$$

for fixed $0 < |\alpha| < 1$ and $t \in \mathbb{C}$. If we consider the complex analytic family $(\mathcal{N}, \mathbb{C}, \pi)$ induced from $(\mathcal{M}, \mathbb{C}, \hat{\omega})$ by the holomorphic map $s \to t = s^2$, then by Theorem1.5.2, we get

$$\rho_s\left(\frac{d}{ds}\right) = \frac{dM_{s^2}}{ds} = \frac{dt}{ds}\frac{dM_t}{dt} = 2s\frac{dM_t}{dt}.$$

This implies that $\rho_s \equiv 0$ because $(\mathcal{M}_U, U, \hat{\omega})$ is trivial, where $U = \mathbb{C}\backslash\{0\}$. On the other hand, $(\mathcal{N}, \mathbb{C}, \pi)$ is not locally trivial. For, $\pi^{-1}(0) = M_0$ and $\pi^{-1}(s) = M_t$ with $t = s^2 \neq 0$ are not biholomorphically equivalent.

Chapter 2

Theorem of Existence

2.1 Introduction

Given a compact complex manifold M, if $(\mathcal{M}, B, \hat{\omega})$ with $0 \in B \subset \mathbb{C}$ is a complex analytic family such that $\hat{\omega}^{-1}(0) = M$, then we have shown in chapter 1 that the infinitesimal deformation $\left(\frac{dM_t}{dt}\right)_{t=0}$ belongs to $H^1(M, \Theta)$, where Θ is the sheaf of germs of holomorphic vector fields over M. Conversely, we would like to ask the following question :

Question : Given a $\theta \in H^1(M, \Theta)$, does there exist a complex analytic family $(\mathcal{M}, B, \hat{\omega})$ with $0 \in B \subset \mathbb{C}$ such that $\hat{\omega}^{-1}(0) = M$ and $\left(\frac{dM_t}{dt}\right)_{t=0} = \theta$?

We study the above question in this chapter.

2.2 Some Facts on the q^{th} Cohomology Group $H^q(M, \Theta)$

Let $\mathcal{U} = \{U_j\}$ be a finite open covering of a compact complex manifold M, and suppose that each U_j is a coordinate polydisk. Let $\mathcal{S}$ be a sheaf over M, $H^1(\mathcal{U}, \mathcal{S})$ be the cohomology group of $\mathcal{U}$ and $H^1(M, \mathcal{S})$ be the cohomology group of M with coefficients in $\mathcal{S}$. Then we have the following basic fact :

Theorem 2.2.1 *If $H^1(U_j, \mathcal{S}) = 0$ for every $U_j \in \mathcal{S}$, then $H^1(\mathcal{U}, \mathcal{S}) = H^1(M, \mathcal{S})$ and $H^2(\mathcal{U}, \mathcal{S}) \hookrightarrow H^2(M, \mathcal{S})$.*

<u>**Notation**</u> :

$$
\begin{aligned}
\mathcal{O} &= \text{sheaf of germs of holomorphic functions over a compact} \\
&\quad\ \text{complex manifold } M \\
\mathcal{A}^{p,q} &= \text{sheaf of germs of } C^\infty \ (p,q)\text{-forms over } M \\
\Gamma(W, \mathcal{S}) &= K\text{-module of all sections of } \mathcal{S} \text{ over } W, \text{where } W \text{ is a domain} \\
&\quad\ \text{of } M \text{ and } K = \mathbb{R} \text{ or } \mathbb{C}
\end{aligned}
$$

Lemma 2.2.2 *Let $U = \{(z^1, \ldots, z^n) \in \mathbb{C}^n \ : \ |z^1| < r, \ldots, |z^n| < r, 0 < r \leq \infty\}$ be a polydisk, and Θ the sheaf of germs of holomorphic vector fields over U. Then $H^q(U, \Theta) = 0$ for $q \geq 1$.*

Proof : For any $v = \sum\limits_{\alpha=1}^{n} v_\alpha \frac{\partial}{\partial z^\alpha} \in \Theta_z$, $z \in U$, we represent it as $v = (v_1, \ldots, v_n)$, $v_\alpha \in \mathcal{O}_z$. Therefore, we may consider

$$H^q(U, \Theta) = \underbrace{H^q(U, \mathcal{O}) \oplus \cdots \oplus H^q(U, \mathcal{O})}_{n \ times}.$$

By Dolbeault's theorem, we have $H^q(U,\mathcal{O}) \cong \Gamma(U,\overline{\partial}\mathcal{A}^{0,q-1})/\overline{\partial}\Gamma(U,\mathcal{A}^{0,q-1})$ for $q \geq 1$. Then, it follows from Dolbeault's lemma that $H^q(U,\Theta) = 0$ for $q \geq 1$. $\square$

From the above results, we get the following lemma :

Lemma 2.2.3 $H^1(\mathcal{U},\Theta) = H^1(M,\Theta)$ *and* $H^2(\mathcal{U},\Theta) \hookrightarrow H^2(M,\Theta)$.

Using the Lie bracket, we put for any two 1-cocycles $\{\theta_{jk}\}$ and $\{\eta_{jk}\}$ in $Z^1(\mathcal{U},\Theta)$,

$$\zeta_{ijk} = \frac{1}{2}\left([\theta_{ij},\eta_j k] + [\eta_{ij},\theta_{jk}]\right). \tag{2.1}$$

Then, it is easy to see that $\{\zeta_{ijk}\}$ forms a 2-cocycle in $Z^2(\mathcal{U},\Theta)$.

Definition 2.2.4 *Let* $\theta,\ \eta \in H^1(M,\Theta)$ *be the cohomology classes of* $\{\theta_{jk}\}$ *and* $\{\eta_{jk}\}$ *respectively. We define the bracket of* θ *and* η *by putting*

$$[\theta,\eta] = \zeta,$$

where ζ *is the cohomology class of* $\{\zeta_{ijk}\}$ *defined by (2.1)*

The bracket is clearly well-defined. In fact, if $\theta_{jk} = \theta_k - \theta_j$, then $2\zeta_{ijk} = [\theta_j + \theta_k, \eta_{jk}]$-$[\theta_i + \theta_k, \eta_{ik}]$+ $[\theta_i + \theta_j, \eta_{ij}]$. Observe that the bracket is bilinear in θ and η, and $[\theta,\eta] = [\eta,\theta]$.

2.3 Obstructions to Deformation

Theorem 2.3.1 *Given an element* θ *in* $H^1(M,\Theta)$. *In order that there may exist a complex analytic family* $(\mathcal{M},B,\hat{\omega})$ *such that* $\hat{\omega}^{-1}(0) = M$, *and that* $\left(\frac{dM_t}{dt}\right)_{t=0} = \theta$, $[\theta,\theta]$ *must be the zero element in* $H^2(M,\Theta)$.

Proof : Let $\theta \in H^1(M, \Theta)$ and suppose that there exists a complex analytic family $(\mathcal{M}, B, \hat{\omega})$ such that $\hat{\omega}^{-1}(0) = M$ and $\left(\frac{dM_t}{dt}\right)_{t=0} = \theta$. Let $\theta(t)$ be the cohomology class of $\{\theta_{jk}(t)\}$. Then by (1.5), we have

$$\theta_{ik}^{\alpha}(z_i, t) = \theta_{ij}^{\alpha}(t) + \sum_{\beta=1}^{n} \frac{\partial z_i^{\alpha}}{\partial z_j^{\beta}} \theta_{jk}^{\beta}(z_j, t), \tag{2.2}$$

where $\theta_{jk}^{\beta}(z_j, t) = \frac{\partial f_{jk}^{\beta}(z_k, t)}{\partial t}$. Differentiating (2.2) in t as functions of $z_j^1, \ldots, z_j^n, t$, we get

$$\begin{aligned} & \theta_{ij} \cdot \theta_{ik}^{\alpha}(z_i, t) + \dot{\theta_{jk}^{\alpha}}(z_i, t) \\ = \;& \theta_{ij}(t) \cdot \theta_{ij}^{\alpha}(z_i, t) + \dot{\theta_{ij}^{\alpha}}(z_i, t) + \theta_{jk}(t) \cdot \theta_{ij}^{\alpha}(z_i, t) + \sum_{\beta} \frac{\partial z_i^{\alpha}}{\partial z_j^{\beta}} \dot{\theta_{jk}^{\beta}}(z_j, t), \end{aligned} \tag{2.3}$$

where $\dot{\theta_{jk}^{\beta}}(z_j, t) = \frac{\partial \theta_{jk}^{\beta}(z_j, t)}{\partial t}$. Using (2.2), we write (2.3) as

$$\begin{aligned} & \dot{\theta_{ij}^{\alpha}}(z_i, t) - \dot{\theta_{ik}^{\alpha}}(z_i, t) + \sum_{\beta=1}^{n} \frac{\partial z_i^{\alpha}}{\partial z_j^{\beta}} \dot{\theta_{jk}^{\beta}}(z_i, t) \\ = \;& \theta_{ij}(t) \cdot \sum_{\beta} \frac{\partial z_i^{\alpha}}{\partial z_j^{\beta}} \theta_{jk}^{\beta}(z_j, t) - \theta_{jk}(t) \cdot \theta_{ij}^{\alpha}(z_i, t). \end{aligned} \tag{2.4}$$

Then, by putting $\dot{\theta}_{ij}(t) = \sum_{\alpha} \dot{\theta}_{ij}^{\alpha}(z_i, t) \frac{\partial}{\partial z_i^{\alpha}}$, we have

$$\dot{\theta}_{ij}(t) - \dot{\theta}_{ik}(t) + \dot{\theta}_{jk}(t) = [\theta_{ij}(t), \theta_{jk}(t)]. \tag{2.5}$$

Therefore, we get $[\theta, \theta] = 0 \in H^2(M, \Theta)$. $\square$

Consequently, if $[\theta, \theta] \neq 0$, the answer of the question stated in section 2.1 is negative. In this sense we call $[\theta, \theta] \in H^2(M, \Theta)$ the obstruction to deformation of M. Observe that by differentiating the fundamental equalities (1.3) m times and putting $t = 0$ for $m = 3, 4, \ldots$, we obtain infinitely many conditions for which θ must satisfy.

2.4 An Elementary Method for Theorem of Existence

Let M^n be a compact complex manifold and $\theta \in H^1(M, \Theta)$. In order to prove that there exists a complex analytic family $(\mathcal{M}, B, \hat{\omega})$ with $0 \in B \subset \mathbb{C}$ such that $\hat{\omega}^{-1}(0) = M$ and $\left(\frac{dM_t}{dt}\right)_{t=0} = \theta$, it suffices to take polydisks $U_j = \{z_j \in \mathbb{C}^n \ : \ |z_j^1| < r, \ldots, |z_j^n| < r, \ r > 0\}$, $j = 1, \ldots, l$, and a sufficiently small disk $\Delta = \{t \in \mathbb{C} \ : \ |t| < r\}$, and then construct a complex structure $\mathcal{M} = \bigcup_{j=1}^{l} U_j \times \Delta$ with defining functions $f_{jk}^{\alpha}(z_k, t)$ satisfying the following conditions :

(i) $M = \bigcup_{j=1}^{l} U_j \times 0 = \bigcup_{j=1}^{l} U_j$ with defining functions $f_{jk}^{\alpha}(z_k, 0)$,

(ii) $\sum\limits_{\alpha=1}^{n} \frac{\partial f_{jk}^{\alpha}(z_k,0)}{\partial t} \frac{\partial}{\partial z_j^{\alpha}} = \theta_{jk}$, where $\{\theta_{jk}\} = \theta$.

Thus, if we expand $f_{jk}^{\alpha}(z_k, t)$ into the power series in t as

$$f_{jk}^{\alpha}(z_k, t) = \sum_{\nu=0}^{\infty} f_{jk|\nu}^{\alpha}(z_k) t^{\nu} \ , \ \alpha = 1, \ldots, n. \tag{2.6}$$

$f_{jk|0}^{\alpha}(z_k)$ must be the defining functions of M and $f_{jk|1}^{\alpha} = \theta_{jk}^{\alpha}$. Now, it remains to determine $f_{jk|\nu}^{\alpha}(z_k)$, $\nu = 2, 3, \ldots$, such that $f_{jk}^{\alpha}(z_k, t)$ define a complex manifold $\mathcal{M} = \bigcup_{j=1}^{l} U_j \times \Delta$. More precisely, $f_{jk}^{\alpha}(z_k, t)$ must be holomorphic in $z_k^1, \ldots, z_k^n, t$ and satisfy the following compatibility conditions on $U_k \times \Delta \cap U_j \times \Delta \neq \emptyset$:

$$f_{ik}^{\alpha}(z_k, t) = f_{ij}^{\alpha}(f_{jk}(z_k, t), \ t) = f_{ij}^{\alpha}(f_{jk}^1(z_k, t), \ldots, f_{jk}^n(z_k, t), \ t) \tag{2.7}$$

<u>**Notation**</u> :

Let $A(t) = \sum\limits_{\nu=0}^{\infty} A_{\nu}(t)$, and $B(t) = \sum\limits_{\nu=0}^{\infty} B_{\nu}(t)$ be polynomials in t. We put

$$A_{\nu} = A_0 + A_1 t + \cdots + A_{\nu} t^{\nu} \quad \text{and} \quad B_{\nu} = B_0 + B_1 t + \cdots + B_{\nu} t^{\nu}.$$

Then, by $A(t) \overset{\nu}{\equiv} B(t)$ we mean $A^\nu(t) = B^\nu(t)$. Using vector notation, we rewrite (2.6) as

$$f_{jk}(z_k, t) = \sum_{\nu=0}^{\infty} f_{jk|\nu}(z_k) t^\nu. \tag{2.8}$$

Therefore, (2.6) is reduced to the system of infinitely many congruences :

$$(\mathbf{I}_\nu) \cdots\cdots \quad f_{ik}^\nu(z_k, t) \overset{\nu}{\equiv} f_{ij}^\nu(f_{jk}^\nu(z_k, t),\ t), \qquad \nu = 1, 2, \ldots$$

As $f_{jk|0}^\alpha(z_k)$ are chosen to be the defining functions of M and $f_{jk|1}^\alpha(z_k) = \theta_{jk}^\alpha(z_j)$, where $z_j = f_{jk}(z_k, 0)$, $(\mathbf{I_1})$ is true. Now, we want to determine $f_{jk|\nu}(z_k)$ for $\nu \geq 2$ by induction on ν such that $(\mathbf{I}_\nu)$ is true for all ν. Assume that

$$f_{jk}^{\nu-1}(z_k, t) = f_{jk|0}(z_k) + \cdots + f_{jk|\nu-1}(z_k) t^{\nu-1} \quad \text{and} \quad (\mathbf{I}_{\nu-1}) \text{ is true.}$$

Then we consider $(\mathbf{I}_\nu)$. Observe that $(\mathbf{I}_\nu)$ is equivalent to

$$f_{ik}^{\nu-1}(z_k, t) + f_{ik|\nu}(z_k) t^\nu \overset{\nu}{\equiv} f_{ij}^{\nu-1}(f_{jk}^{\nu-1}(z_k, t),\ t) + \sum_{\beta=1}^{n} \frac{\partial z_i}{\partial z_j^\beta} f_{jk|\nu}^\beta(z_k) t^\nu + f_{ij|\nu}(z_j) t^\nu.$$

That is

$$f_{ik}^{\nu-1}(z_k, t) - f_{ij}^{\nu-1}(f_{jk}^{\nu-1}(z_k, t), t) \overset{\nu}{\equiv} \left(\sum_{\beta=1}^{n} \frac{\partial z_i}{\partial z_j^\beta} f_{jk|\nu}^\beta(z_k) - f_{ik|\nu}(z_k) + f_{ij|\nu}(z_j) \right) t^\nu,$$

where $z_i = f_{ij|0}(z_j)$. Let $\Gamma_{ijk|\nu}(z_k)$ be the coefficient of t^ν in the left hand side of the above congruence. Then by assumption, we have

$$f_{ik}^{\nu-1}(z_k, t) - f_{ij}^{\nu-1}(f_{jk}^{\nu-1}(z_k, t), t) \overset{\nu}{\equiv} \Gamma_{ijk|\nu}(z_k) t^\nu. \tag{2.9}$$

Hence, $(\mathbf{I}_\nu)$ is reduced to the equality :

$$\Gamma_{ijk|\nu}(z_k) = \sum_{\beta=1}^{n} \frac{\partial z_i}{\partial z_j^\beta} f_{jk|\nu}^\beta(z_k) - f_{ik|\nu}(z_k) + f_{ij|\nu}(z_j) \tag{2.10}$$

Writing in terms of the components, (2.10) becomes

$$\Gamma_{ijk|\nu}^\alpha(z_k) = \sum_{\beta=1}^{n} \frac{\partial z_i}{\partial z_j^\beta} f_{jk|\nu}^\beta(z_k) - f_{ik|\nu}^\alpha(z_k) + f_{ij|\nu}^\alpha(z_j).$$

Using vector field notation, we get

$$\sum_\alpha \Gamma_{ijk|\nu}^\alpha(z_k)\frac{\partial}{\partial z_i^\alpha} = \sum_\beta f_{jk|\nu}^\beta(z_k)\frac{\partial}{\partial z_j^\beta} - \sum_\alpha f_{ik|\nu}^\alpha(z_k)\frac{\partial}{\partial z_i^\alpha} + \sum_\alpha f_{ij|\nu}^\alpha(z_j)\frac{\partial}{\partial z_i^\alpha}.$$

Putting

$$\Gamma_{ijk|\nu} = \sum_\alpha \Gamma_{ijk|\nu}^\alpha(z_k)\frac{\partial}{\partial z_i^\alpha} \quad \text{and} \quad f_{jk|\nu} = \sum_\beta f_{jk|\nu}^\beta(z_k)\frac{\partial}{\partial z_j^\beta},$$

we obtain

$$\Gamma_{ijk|\nu} = f_{jk|\nu} - f_{ik|\nu} + f_{ij|\nu}. \tag{2.11}$$

Now, we extend the definitions of 1-cochains, 2-cocycles, etc., as follow :

Let $\mathcal{S}$ be an arbitrary sheaf over M and $\mathcal{U} = \{U_j\}$ the finite cover of M. Then the set $\hat{c}^1 = \{\sigma_{jk}\}$ of sections of $\Gamma(U_j \cap U_k,\ \mathcal{S})$ is called a 1-cochain on $\mathcal{U}$ if $\sigma_{jj} = 0$ for every j. Similarly, the set $\hat{c}^2 = \{\sigma_{ijk}\}$ of sections $\sigma_{ijk} \in \Gamma(U_i \cap U_j \cap U_k,\ \mathcal{S})$ is called a 2-cocycle on $\mathcal{U}$ if

$$\begin{cases} \sigma_{iik} = \sigma_{ikk} = 0 \\ \sigma_{ijk} - \sigma_{hjk} + \sigma_{hik} - \sigma_{hij} = 0 \quad \text{on each } U_h \cap U_i \cap U_j \cap U_k \neq \emptyset. \end{cases}$$

Next, we define the coboundary of a 1-cochain $\hat{c}^1 = \{\sigma_{jk}\}$ as

$$\delta\hat{c}^1 = \{\tau_{ijk}\},\ \tau_{ijk} = \sigma_{jk} - \sigma_{ik} + \sigma_{ij}.$$

Then, it is clear that the coboundary $\delta\hat{c}^1$ is a 2-cocycle. Let $\hat{C}^1(\mathcal{U},\mathcal{S})$ be the abelian group of all 1-cochains $\hat{c}^1$ and $\hat{Z}^2(\mathcal{U},\mathcal{S})$ that of all 2-cocycles $\hat{c}^2$. Then the quotient group

$$\hat{H}^2(\mathcal{U},\mathcal{S}) = \hat{Z}^2(\mathcal{U},\mathcal{S})/\delta\hat{C}^1(\mathcal{U},\mathcal{S})$$

is called the 2-dimensional cohomology group of $\mathcal{U}$ with coefficients in $\mathcal{S}$. With this notation, (2.11) is written as

$$\{\Gamma_{ijk|\nu}\} = \delta\{f_{jk|\nu}\}. \tag{2.12}$$

Lemma 2.4.1 *$\{\Gamma_{ijk|\nu}\}$ is a 2-cocycle on $\mathcal{U}$, where $\Gamma_{ijk|\nu} = \sum\limits_{\alpha}\Gamma^{\alpha}_{ijk|\nu}\frac{\partial}{\partial z_i^{\alpha}}$.*

Proof : It suffices to prove that

$$\sum_{\alpha}\frac{\partial z_h}{\partial z_i^{\alpha}}\Gamma^{\alpha}_{ijk|\nu} - \Gamma_{hjk|\nu}(z_k) + \Gamma_{hik|\nu}(z_k) - \Gamma_{hij|\nu} = 0,$$

where $z_h = (z_h^1,\ldots,z_h^n)$ and $z_h = f_{hk|0}(z_k)$.
For simplicity, we write $\Gamma_{hjk|\nu}(z_k)$, $f_{hk}^{\nu-1}(z_k,t)$ as $\Gamma_{hjk|\nu}$, $f_{hk}^{\nu-1}(t)$ respectively. By (2.9), we have

$$\begin{aligned}
\Gamma_{ijk|\nu}t^{\nu} &\overset{\nu}{\equiv} f_{ik}^{\nu-1}(t) - f_{ij}^{\nu-1}(f_{jk}^{\nu-1}(t),\ t), & (2.13)\\
-\Gamma_{hjk|\nu}t^{\nu} + \Gamma_{hik|\nu}t^{\nu} &\overset{\nu}{\equiv} f_{hj}^{\nu-1}(f_{jk}^{\nu-1}(t),\ t) - f_{hi}^{\nu-1}(f_{ik}^{\nu-1}(t),\ t), & (2.14)\\
\Gamma_{hij|\nu}(z_j)t^{\nu} &\overset{\nu}{\equiv} f_{hj}^{\nu-1}(z_j,t) - f_{hi}^{\nu-1}(f_{ij}^{\nu-1}(z_j,t),\ t). & (2.15)
\end{aligned}$$

Putting $z_j = f_{jk}^{\nu-1}(t)$ into (2.15), we get

$$\Gamma_{hij|\nu}t^{\nu} \overset{\nu}{\equiv} f_{hj}^{\nu-1}(f_{jk}^{\nu-1}(t),\ t) - f_{hi}^{\nu-1}(f_{ij}^{\nu-1}(f_{jk}^{\nu-1}(t),\ t),\ t). \tag{2.16}$$

By (2.13), we have

$$\begin{aligned} f_{hi}^{\nu-1}(f_{ik}^{\nu-1}(t),\ t) &\overset{\nu}{\equiv} f_{hi}^{\nu-1}(f_{ij}^{\nu-1}(f_{jk}^{\nu-1}(t),\ t)+\Gamma_{ijk|\nu}t^{\nu},\ t) \\ &\overset{\nu}{\equiv} f_{hi}^{\nu-1}(f_{ij}^{\nu-1}(f_{jk}^{\nu-1}(t),\ t),\ t)+\sum_{\alpha=1}^{n}\tfrac{\partial z_h}{\partial z_i^{\alpha}}\Gamma_{ijk|\nu}t^{\nu}. \end{aligned} \tag{2.17}$$

Substituting (2.17) into (2.16), we get

$$\Gamma_{hij|\nu}t^{\nu} \overset{\nu}{\equiv} f_{hj}^{\nu-1}(f_{jk}^{\nu-1}(t),\ t) - f_{hi}^{\nu-1}(f_{ik}^{\nu-1}(t),\ t) + \sum_{\alpha=1}^{n}\frac{\partial z_h}{\partial z_i^{\alpha}}\Gamma_{ijk|\nu}t^{\nu}. \tag{2.18}$$

Hence, by subtracting (2.18) from (2.14), we get

$$-\Gamma_{hjk|\nu}t^{\nu}+\Gamma_{hik|\nu}t^{\nu}-\Gamma_{hij|\nu}t^{\nu} = -\sum_{\alpha=1}^{n}\frac{\partial z_h}{\partial z_i^{\alpha}}\Gamma_{ijk|\nu}t^{\nu}.$$

$\square$

Let $\Gamma_{\nu} \in \hat{H}^2(\mathcal{U},\Theta)$ be the cohomology class of the 2-cocycle $\{\Gamma_{ijk|\nu}\} \in \hat{Z}^2(\mathcal{U},\Theta)$. Then (2.12) implies that if $\Gamma_{\nu} \neq 0$, then we cannot determine $\{f_{jk|\nu}\}$ such that $(\mathbf{I}_{\nu})$ holds. In this sense, we call Γ_{ν} the obstruction to deformation of M. As

$$\Gamma_{ijk|\nu} = f_{ik|0}(z_k) - f_{ij|0}(f_{jk|0}(z_k)) = 0,$$

we have $\Gamma_1 = 0$. Thus we call Γ_2 the first obstruction, and $\Gamma_{\nu+1}$ the ν^{th} obstruction. If $\Gamma_{\nu} = 0$, then in order to make $(\mathbf{I}_{\nu})$ holds, we choose a 1-cochain $\{\sigma_{jk}\} \in \hat{C}^1(\mathcal{U},\Theta)$ with $\{\Gamma_{ijk|\nu}\} = \delta\{\sigma_{jk}\}$, and put $f_{jk|\nu} = \sigma_{jk}$. Since

$$\Gamma_{ijk|\nu}t^{\nu} \overset{\nu}{\equiv} f_{ik}^{\nu-1}(z_k, t) - f_{ij}^{\nu-1}(f_{jk}^{\nu-1}(z_k, t),\ t),$$

Γ_{ν} is not defined unless $\Gamma_2 = 0, \ldots, \Gamma_{\nu-1} = 0$. Moreover, even if $\Gamma_2 = 0, \ldots, \Gamma_{\nu-1} = 0$, Γ_{ν} depends in general on the choice of $\{f_{jk}^{\nu-1}(z_k, t)\}$. Thus, it may be that $\Gamma_\nu \neq 0$ for one choice of $\{f_{jk}^{\nu-1}(z_k,t)\}$, and $\Gamma_\nu = 0$ for another. Therefore, it is difficult to see whether we can determine $\{f_{jk|\nu}\}$ such that $(\mathbf{I}_\nu)$ hold for all ν.

Obviously, if $\hat{H}^2(\mathcal{U},\Theta) = 0$, then we can construct a formal power series in t:

$$f_{jk}(z_k,t) = \sum_{\nu=0}^{\infty} f_{jk|\nu}(z_k)t^\nu$$

satisfying the compatibility conditions

$$f_{ik}(z_k,t) = f_{ij}(f_{jk}(z_k,t),\ t).$$

Note that if all $f_{jk}^{\alpha}(z_k,t)$ converge for $|t| < r$, then $f_{jk}^{\alpha}(z_k,\ t)$ are holomorphic functions on $(U_j \cap U_k) \times \Delta$, where $\Delta = \{t \in \mathbb{C} \ :\ |t| < r\}$. Thus, it remains to show that we can choose $f_{jk}^{\alpha}(z_k,t)$ such that they are all convergent on $(U_j \cap U_k) \times \Delta$. However, it turns out that this is difficult, indeed insurmountable, so that the elementary method fails, yet, we do get some useful facts, as follows.

Lemma 2.4.2 *If $H^2(M,\Theta) = 0$, then $\hat{H}^2(\mathcal{U},\Theta) = 0$.*

Proof : It suffices to prove that for any 2-cocycle $\{\theta_{ijk}\} \in \hat{Z}^2(\mathcal{U},\Theta)$, there exists $\{\theta_{jk}\} \in \hat{C}^1(\mathcal{U},\Theta)$ such that $\delta\{\theta_{jk}\} = \{\theta_{ijk}\}$. Let $\{\rho_j\}$ be a partition of unity subordinate to $\mathcal{U} = \{U_j\}$. By the definition of a 2-cocycle, we have

$$\theta_{ijk} = \theta_{hjk} - \theta_{hik} + \theta_{hij}.$$

Multiplying both sides by ρ_h, and taking the summation with respect to h, we get

$$\theta_{ijk} = \xi_{jk} - \xi_{ik} + \xi_{ij}, \text{ where } \xi_{jk} = \sum_h \rho_h \theta_{hjk}.$$

Then each ξ_{jk} is a C^∞ vector field on $U_j \cap U_k \neq \emptyset$. Since $\overline{\partial}\theta_{ijk} = 0$, we have

$$\overline{\partial}\xi_{jk} = \bar{\partial}\xi_{ik} - \bar{\partial}\xi_{ij}.$$

Putting $\psi_k = \sum_i \rho_i \bar{\partial}\xi_{ik}$, we get

$$\bar{\partial}\xi_{jk} = \psi_k - \psi_j.$$

Clearly, ψ_k is a C^∞ vector $(0,1)$-form on U_k and $\bar{\partial}\psi_k = \bar{\partial}\psi_j$ on $U_k \cap U_j$. Putting

$$\varphi = \bar{\partial}\psi_j \text{ on each } U_j,$$

we get an element φ in $\Gamma(M,\ \bar{\partial}\mathcal{A}^{0,1}(T(M)))$. By assumption, we have

$$H^2(M,\Theta) = \Gamma(M,\ \bar{\partial}\mathcal{A}^{0,1}(T(M)))/\bar{\partial}\Gamma(M,\ \mathcal{A}^{0,1}(T(M))) = 0.$$

This implies that there exists $\psi \in \Gamma(M,\ \mathcal{A}^{0,1}(T(M)))$ such that $\varphi = \bar{\partial}\psi$. Thus, we have

$$\bar{\partial}(\psi_j - \psi) = 0 \text{ on each } U_j.$$

Then, by Dolbeault's lemma, there exists a C^∞ vector field η_j on U_j such that

$$\psi_j - \psi = \bar{\partial}\eta_j.$$

Hence we have

$$\bar{\partial}\xi_{jk} = \psi_k - \psi_j = \bar{\partial}\eta_k - \bar{\partial}\eta_j.$$

This implies that

$$\bar{\partial}(\xi_{jk} - \eta_k + \eta_j) = 0.$$

Then, putting

$$\theta_{jk} = \xi_{jk} - \eta_k + \eta_j,$$

we get a holomorphic vector field θ_{jk} on $U_j \cap U_k$. Since $\theta_{hjj} = 0$, we have $\xi_{jj} = 0$, and thus $\theta_{jj} = 0$. Hence $\{\theta_{jk}\} \in \hat{C}^1(\mathcal{U}, \Theta)$. Further, it is clear that

$$\theta_{jk} - \theta_{ik} + \theta_{ij} = \xi_{jk} - \xi_{ik} + \xi_{ij} = \theta_{ijk}.$$

□

Consequently, if $H^2(M, \Theta) = 0$, then $f_{ik}(z_k,\ t) = f_{ij}(f_{jk}(z_k, t),\ t)$ have formal power series solutions and thus it is reasonable to hope that for any $\theta \in H^1(M, \Theta)$, there exists a complex analytic family $(\mathcal{M}, \Delta, \hat{\omega})$ such that

$$\hat{\omega}^{-1}(0) = M, \quad \left(\frac{dM_t}{dt}\right)_{t=0} = \theta,$$

where $M_t = \hat{\omega}^{-1}(t)$.

Lemma 2.4.3 $[\theta, \theta] = 2\Gamma_2$.

Proof : Observe that

$$H^2(\mathcal{U}, \Theta) \hookrightarrow \hat{H}^2(\mathcal{U}, \Theta).$$

Thus in order to prove the lemma, it suffices to show that there exists a 1-cochain $\{\tau_{jk}\} \in \hat{C}^1(\mathcal{U}, \Theta)$ such that

$$2\Gamma_{ijk|2} - [\theta_{ij}, \theta_{jk}] = \tau_{jk} - \tau_{ik} + \tau_{ij}. \tag{2.19}$$

As

$$\Gamma_{ijk|2}(z_k)t^2 \overset{2}{\equiv} f^1_{ik}(z_k,\ t) - f^1_{ij}(f^1_{jk}(z_k, t),\ t),$$

we have

$$\Gamma_{ijk|2} = -\frac{1}{2}\sum_{\beta,\gamma=1}^{n} \frac{\partial^2 z_i}{\partial z_j^\beta \partial z_j^\gamma} f^\beta_{jk|1} f^\gamma_{jk|1} - \sum_{\beta=1}^{n} \frac{\partial f_{ij|1}}{\partial z_j^\beta} f^\beta_{jk|1}.$$

Since $f_{jk|1} = \theta_{jk} = (\theta^1_{jk}, \dots, \theta^n_{jk})$, we get

$$2\Gamma_{ijk|2} = -\sum_{\beta,\gamma} \theta^\gamma_{jk}\theta^\beta_{jk} \frac{\partial^2 z_i}{\partial z^\gamma_j \partial z^\beta_j} - 2\theta_{jk} \cdot \theta_{ij},$$

where $\theta_{jk} = \sum\limits_\beta \theta^\beta_{jk} \frac{\partial}{\partial z^\beta_j}$. Then, by product rule, we obtain

$$2\Gamma_{ijk|2} = -\theta_{jk}\left(\sum_\beta \frac{\partial z_i}{\partial z^\beta_j}\theta^\beta_{jk}\right) + \sum_\beta \frac{\partial z_i}{\partial z^\beta_j}\theta_{jk} \cdot \theta^\beta_{jk} - 2\theta_{jk} \cdot \theta_{ij}. \tag{2.20}$$

As $\{\theta_{jk}\} \in Z^2(\mathcal{U}, \Theta)$, we have

$$\sum_\beta \frac{\partial z_i}{\partial z^\beta_j}\theta^\beta_{jk} = \theta_{ik} - \theta_{ij}.$$

Putting it into (2.20), we get

$$2\Gamma_{ijk|2} = \sum_\beta \frac{\partial z_i}{\partial z^\beta_j}\theta_{jk} \cdot \theta^\beta_{jk} - \theta_{jk} \cdot \theta_{ik} - \theta_{jk} \cdot \theta_{ij}.$$

In other words, we have

$$2\Gamma^\alpha_{ijk|2} = \sum_\beta \frac{\partial z^\alpha_i}{\partial z^\beta_j}\theta_{jk} \cdot \theta^\beta_{jk} - \theta_{jk} \cdot \theta^\alpha_{ik} - \theta_{jk} \cdot \theta^\alpha_{ij},\ \alpha = 1, \dots, n.$$

In terms of holomorphic vector fields, we obtain

$$2\Gamma_{ijk|2} = 2\sum_\alpha \Gamma^\alpha_{ijk|2} \frac{\partial}{\partial z^\alpha_i} = \theta_{jk} \cdot \theta_{jk} - \theta_{jk} \cdot \theta_{ik} - \theta_{jk} \cdot \theta_{ij}.$$

Since

$$[\theta_{ij}, \theta_{jk}] = [\theta_{ij}, \theta_{ik}] = \theta_{ij} \cdot \theta_{ik} - \theta_{ik} \cdot \theta_{ij},$$

we have

$$2\Gamma_{ijk|2} - [\theta_{ij}, \theta_{jk}] = \theta_{jk} \cdot \theta_{jk} - \theta_{ik} \cdot \theta_{ik} + \theta_{ij} \cdot \theta_{ij}.$$

Putting

$$\tau_{jk} = \theta_{jk} \cdot \theta_{jk} = \sum_\beta \left(\theta_{jk} \cdot \theta^\beta_{jk}\right) \frac{\partial}{\partial z^\beta_j},$$

we obtain (2.19). $\square$

2.5 Proof of Theorem of Existence

Theorem 2.5.1 (Theorem of Existence)[7] *Let M be a compact complex manifold and $H^2(M,\Theta) = 0$. Then there exists a complex analytic family $(\mathcal{M}, B, \hat{\omega})$ with $0 \in B \subset \mathbb{C}^m$ satisfying the following conditions :*
(i) $\hat{\omega}^{-1}(0) = M$,
(ii) *The linear map $\rho_0 : T_0(B) \to H^1(M,\Theta)$ given by*

$$\frac{\partial}{\partial t} \to \left(\frac{\partial M_t}{\partial t}\right)_{t=0}$$

is an isomorphism, where $M_t = \hat{\omega}^{-1}(t)$.

Before we prove the theorem of existence, we first try to explain the idea behind the proof.

By abuse of notation, we denote the underlying differentiable manifold of the compact complex manifold M also by M. Suppose that there exists a complex analytic family $(\mathcal{M}, B, \hat{\omega})$ with $0 \in B \subset \mathbb{C}^m$ such that $\hat{\omega}^{-1}(0) = M$. Then, if we take a sufficiently small Δ with $0 \in \Delta \subset B$, by Theorem(1.1.3), there is a diffeomorphism Ψ of $M \times \Delta$ onto $\mathcal{M}_\Delta = \cup_{j=1}^{l} \times \Delta$ such that
(i) $\Psi(z,0) = z$ for all $z \in M$ and
(ii) $\hat{\omega} \circ \Psi(z,t) = t$, where $t \in \Delta$.
Put

$$\Phi(z,t) = (\zeta_j^1(z,t), \ldots, \zeta_j^n(z,t), t_1, \ldots, t_m) \quad \text{for} \quad \Phi(z,t) \in U_j \times \Delta.$$

Then, $\mathcal{M}_\Delta$ may be considered as a complex manifold with the complex structure defined on the differentiable manifold $M \times \Delta$ by the system of local complex coordinates :

$$\{(\zeta_j, t) = (\zeta_j^1(z,t), \ldots, \zeta_j^n(z,t), t_1, \ldots, t_m) \;:\; 1 \le j \le l\}.$$

Clearly, M_t is the complex structure of the differentiable manifold M defined by the system of local complex coordinates :

$$\{z \to \zeta_j(z,t) \;:\; 1 \le j \le l\}.$$

Let $(z^1, \ldots, z^n)$ be an arbitrary local complex coordinates of $z \in M$. Then $\zeta_j^\alpha(z,t) = \zeta_j^\alpha(z^1, \ldots, z^n, t_1, \ldots, t_m)$ are C^∞ functions of $z^1, \ldots, z^n, t_1, \ldots, t_m$. Since

$$\begin{cases} \det(\partial_\lambda \zeta_j^\alpha(z,0)) & \neq & 0, \\ \det\left(\frac{\partial \zeta_j^\alpha}{\partial \zeta_k^\beta}\right) & \neq & 0, \\ \det\left(\partial_\lambda \zeta_j^\alpha(z,t)\right) & = & \det\left(\frac{\partial \zeta_j^\alpha}{\partial \zeta_k^\beta}\right) \cdot \det\left(\partial_\lambda \zeta_k^\beta(z,t)\right), \end{cases}$$

where $\partial_\lambda = \frac{\partial}{\partial z^\lambda}$, we have

$$\det\left(\partial_\lambda \zeta_j^\alpha(z,t)\right) \neq 0, \text{ provided that } \Delta \text{ is sufficiently small.} \tag{2.21}$$

<u>**Notation**</u> :

$T = T(M)$	:	tangent bundle of M.
$\mathcal{A}(T)$	:	sheaf of germs of C^∞ vector fields on M.
$\mathcal{A}^{p,q}(T)$	:	sheaf of germs of C^∞ vector (p,q) – forms on M.
$\mathcal{L}^{0,q}(T)$	:	linear space of C^∞ sections of $T \otimes \bigwedge^q \overline{T^*}$ over M.
$\mathcal{L}^{0,q}_{\overline{\partial}}$	:	linear subspace of $\mathcal{L}^{0,q}(T)$ consisting of all $\overline{\partial}$ – closed vector $(0,q)$ – forms.

Note that $\mathcal{L}_{\bar\partial}^{0,q}(T) = \Gamma(M, \bar\partial A^{0,q}(T))$. Now we consider

$$\bar\partial \zeta_j^\alpha(z,t) = \sum_\nu \bar\partial_\nu \zeta_j^\alpha(z,t) d\bar z^\nu, \text{ where } \bar\partial_\nu = \frac{\partial}{\partial \bar z^\nu}.$$

By (2.21), there is a unique $(0,1)$-form

$$\varphi_j^\lambda(z,t) = \sum_\nu \varphi_{j\bar\nu}^\lambda(z,t) d\bar z^\nu, \ \lambda = 1, \dots, n, \text{ such that}$$

$$\bar\partial \zeta_j^\alpha(z,t) = \sum_\lambda \varphi_j^\lambda(z,t) \partial_\lambda \zeta_j^\alpha(z,t), \qquad \alpha = 1, \dots, n. \tag{2.22}$$

Lemma 2.5.2 *For any $t \in \Delta$, there is a C^∞ vector $(0,1)$-form $\varphi(z,t)$ on M such that*

$$\varphi(t) = \varphi(z,t) = \sum_{\lambda=1}^n \varphi_j^\lambda(z,t)\partial_\lambda \text{ on } \mathcal{U}_j = \Psi^{-1}(U_j \times \Delta) \quad \text{and} \quad \varphi(0) = 0.$$

Proof : It suffices to prove that

$$\sum_\lambda \varphi_j^\lambda(z,t)\partial_\lambda = \sum_\lambda \varphi_k^\lambda(z,t)\partial_\lambda \quad \text{on} \quad \mathcal{U}_j \cap \mathcal{U}_k. \tag{2.23}$$

Applying $\bar\partial$ to the following fundamental equality :

$$\zeta_j^\alpha(z,t) = f_{jk}^\alpha(\zeta_k(z,t), \ t), \tag{2.24}$$

we get

$$\bar\partial \zeta_j^\alpha(z,t) = \sum_\beta \frac{\partial \zeta_j^\alpha}{\partial \zeta_k^\beta} \bar\partial \zeta_k^\beta(z,t), \text{ where } \zeta_j^\alpha = f_{jk}^\alpha(\zeta_k, t).$$

Then, by (2.22), we have

$$\sum_\lambda \varphi_j^\lambda(z,t)\partial_\lambda \zeta_j^\alpha(z,t) = \sum_\beta \frac{\partial \zeta_j^\alpha}{\partial \zeta_k^\beta} \sum_\lambda \varphi_k^\lambda(z,t)\partial_\lambda \zeta_k^\beta(z,t). \tag{2.25}$$

On the other hand, applying ∂ to (2.24), we get

$$\partial_\lambda \zeta_j^\alpha(z,t) = \sum_\beta \frac{\partial \zeta_j^\alpha}{\partial \zeta_k^\beta} \partial_\lambda \zeta_k^\beta(z,t).$$

Hence, (2.25) can be rewritten as

$$\sum_\lambda \varphi_j^\lambda(z,t)\partial_\lambda \zeta_j^\alpha(z,t) = \sum_\lambda \varphi_k^\lambda(z,t)\partial_\lambda \zeta_j^\alpha(z,t).$$

Since $\det(\partial_\lambda \zeta_j^\alpha(z,t)) \neq 0$, we have

$$\varphi_j^\lambda(z,t) = \varphi_k^\lambda(z,t) \quad \text{on} \quad \mathcal{U}_j \cap \mathcal{U}_k \cap U \times \Delta \neq \emptyset,$$

where U is the domain of the local coordinates $(z^1,\ldots,z^n)$. As $(z^1,\ldots,z^n)$ are arbitrary local complex coordinates of M, we get (2.23). Moreover, since $\bar\partial \zeta_j^\alpha(z,0) = 0,\ \alpha = 1,\ldots,n$, we have $\varphi(0) = 0$. □

In terms of local coordinates $(z^1,\ldots,z^n)$, we have

$$\varphi(t) = \varphi(z,t) = \sum_{\lambda=1}^n \varphi^\lambda(z,t)\partial_\lambda, \tag{2.26}$$

where $\varphi^\lambda(z,t) = \sum_\nu \varphi_{\bar\nu}^\lambda(z,t) d\bar z^\nu$, and

$$\bar\partial \zeta_j^\alpha(z,t) = \sum_\lambda \varphi^\lambda(z,t)\partial_\lambda \zeta_j^\alpha(z,t), \qquad \alpha = 1,\ldots,n. \tag{2.27}$$

Now, we consider $\varphi(t)$ as a differential operator and rewrite (2.27) as

$$(\bar\partial - \varphi(t))\zeta_j^\alpha(z,t) = 0, \qquad \alpha = 1,\ldots,n. \tag{2.28}$$

Theorem 2.5.3 *Let* $\varphi = \sum_{\lambda=1}^n \sum_{\nu=1}^n \varphi_{\bar\nu}^\lambda d\bar z^\nu \partial_\lambda$ *be a* C^∞ *vector* $(0,1)$*−form defined on a domain* U *of* M. *Suppose that the system of the partial differential equations*

$$L_\nu f = (\bar\partial_\nu - \sum_\lambda \varphi_{\bar\nu}^\lambda \partial_\lambda) f = 0, \qquad \nu = 1,\ldots,n,$$

has n linearly independent C^∞ solutions

$$f = \zeta^\alpha = \zeta^\alpha(z) \ \ on\ \ U, \quad \alpha = 1, \ldots, n.$$

Then for any C^∞ function f on $U \subset M$, we have

$$(\bar\partial - \varphi)f = 0 \quad \text{if and only if} \quad \frac{\partial f}{\partial \overline{\zeta^\alpha}} = 0, \ \alpha = 1, \ldots, n.$$

Here the solutions $\zeta^1, \ldots, \zeta^n$ are said to be linearly independent if

$$det\frac{\partial(\zeta^1, \ldots, \zeta^n, \bar{\zeta^1}, \ldots, \bar{\zeta^n})}{\partial(z^1, \ldots, z^n, \bar{z^1}, \ldots, \bar{z^n})} \neq 0.$$

Proof : By assumption, we have

$$\det\frac{\partial(\zeta^1, \ldots, \zeta^n, \bar{\zeta^1}, \ldots, \bar{\zeta^n})}{\partial(z^1, \ldots, z^n, \bar{z^1}, \ldots, \bar{z^n})} = \det(\delta_\nu^\lambda - \sum_\mu \varphi_{\bar\nu}^\mu \overline{\varphi_{\bar\mu}^\lambda})|\det(\partial_\lambda \zeta^\alpha)|^2 \neq 0. \quad (2.29)$$

This implies that

$$\det(\delta_\nu^\lambda - \sum_\mu \varphi_{\bar\nu}^\mu \overline{\varphi_{\bar\mu}^\lambda}) \neq 0 \quad \text{and} \quad \det(\partial_\lambda \zeta^\alpha) \neq 0.$$

Thus $(\zeta^1(z), \ldots, \zeta^n(z))$ is a complex coordinates on $U \subset M$ if M is considered as a differentiable manifold. Hence, we may consider $f = f(\zeta^1, \ldots, \zeta^n)$ as a C^∞ function of $\zeta^1 = \zeta^1(z), \ldots, \zeta^n = \zeta^n(z)$ on M. Then

$$(\bar\partial - \varphi)f(\zeta^1(z), \ldots, \zeta^n(z)) = \sum_\alpha (\bar\partial - \varphi)\zeta^\alpha(z)\frac{\partial f}{\partial \zeta^\alpha} + \sum_\alpha (\bar\partial - \varphi)\overline{\zeta^\alpha(z)}\frac{\partial f}{\partial \overline{\zeta^\alpha}}.$$

As $L_\nu \zeta^\alpha(z) = 0$ for $\nu = 1, \ldots, n$, we have $(\bar\partial - \varphi)\zeta^\alpha(z) = 0$. Thus we obtain

$$(\bar\partial - \varphi)f = \sum_\alpha \sum_\nu (\overline{\partial_\nu \zeta^\alpha} - \sum_\mu \varphi_{\bar\nu}^\mu \partial_\mu \overline{\zeta^\alpha})d\overline{z^\nu}\frac{\partial f}{\partial \overline{\zeta^\alpha}}.$$

Since $\bar{\partial}_\mu \zeta^\alpha = \sum_\lambda \varphi^\lambda_{\bar{\mu}} \partial_\lambda \zeta^\alpha$, we have $\partial_\mu \bar{\zeta^\alpha} = \sum_\lambda \bar{\varphi^\lambda_{\bar{\mu}}} \overline{\partial_\lambda \zeta^\alpha}$. Hence we get

$$\begin{aligned}(\bar{\partial} - \varphi)f &= \sum_\alpha \sum_\nu (\overline{\partial_\nu \zeta^\alpha} - \sum_\mu \sum_\lambda \varphi^\mu_{\bar{\nu}} \overline{\varphi^\lambda_{\bar{\mu}} \partial_\lambda \zeta^\alpha}) d\bar{z}^\nu \frac{\partial f}{\partial \bar{\zeta^\alpha}} \\ &= \sum_\alpha \sum_\nu \sum_\lambda (\delta^\lambda_\nu - \sum_\mu \varphi^\mu_{\bar{\nu}} \bar{\varphi^\lambda_{\bar{\mu}}}) \overline{\partial_\lambda \zeta^\alpha} d\bar{z}^\nu \frac{\partial f}{\partial \bar{\zeta^\alpha}}.\end{aligned}$$

Since $\det(\delta^\lambda_\nu - \sum_\mu \varphi^\mu_{\bar{\nu}}(z) \overline{\varphi^\lambda_{\bar{\mu}}(z)}$ is invariant under the change of local coordinates $(z^1, \dots, z^n)$, it is a differentiable function on M and is nonzero by (2.29). Moreover, we have $\det(\partial_\lambda \zeta^\alpha) \neq 0$ by (2.29). Therefore,

$$(\bar{\partial} - \varphi)f = 0 \quad \text{if and only if} \quad \frac{\partial f}{\partial \bar{\zeta^\alpha}} = 0 \text{ for } \alpha = 1, \dots, n.$$

In other words, $(\bar{\partial} - \varphi)f = 0$ if and only if f is holomorphic in $\zeta^1, \dots, \zeta^n$. □

Corollary 2.5.4 *If we take a sufficiently small polydisk Δ, then for any $t \in \Delta$, a local C^∞ function f on M is holomorphic with respect to the complex structure M_t if and only if it satisfies the equation*

$$(\bar{\partial} - \varphi(t))f = 0. \tag{2.30}$$

Proof : For any $t \in \Delta$, by (2.28), we have

$$(\bar{\partial}_\nu - \sum_\lambda \varphi^\lambda_{\bar{\nu}}(z,t) \partial_\lambda) \zeta^\alpha_j(z,t) = 0 \quad \text{for} \quad \nu = 1, \dots, n.$$

Moreover, as $\varphi(z, 0) = 0$, we have

$$\det(\delta^\lambda_\nu - \sum_\mu \varphi^\mu_{\bar{\nu}}(z,t) \overline{\varphi^\lambda_{\bar{\mu}}(z,t)}) \neq 0 \text{ provided that } \Delta \text{ is sufficiently small.}$$

□

This corollary implies that the deformation M_t of the complex structure on M is represented by the vector $(0,1)-$form

$$\varphi(t) = \sum_{\lambda} \varphi^{\lambda}(z,t)\partial_{\lambda} = \sum_{\lambda,\nu} \varphi^{\lambda}_{\bar{\nu}}(z,t)d\bar{z^{\nu}}\partial_{\lambda} \quad \text{on} \quad M.$$

Definition 2.5.5 *For any*

$$\begin{aligned} \varphi = & \sum_{\lambda} \varphi^{\lambda}\partial_{\lambda} \in \mathcal{L}^{0,p}(T) \\ \textit{and} \quad \psi = & \sum_{\lambda} \psi^{\lambda}\partial_{\lambda} \in \mathcal{L}^{0,q}(T), \end{aligned}$$

where $\varphi^{\lambda} = \frac{1}{q!} \sum_{\nu_1,\dots,\nu_p} \varphi^{\lambda}_{\bar{\nu_1}\dots\bar{\nu_p}}(z)dz^{\bar{\nu}_1} \wedge \cdots \wedge dz^{\bar{\nu}_p}$, *we define their bracket by*

$$[\varphi,\psi] = \sum_{\lambda,\mu} \left(\varphi^{\mu} \wedge \partial_{\mu}\psi^{\lambda} - (-1)^{pq}\psi^{\mu} \wedge \partial_{\mu}\varphi^{\lambda}\right)\partial_{\lambda},$$

where $\partial_{\mu}\psi^{\lambda} = \frac{1}{q!} \sum_{\nu_1,\cdots,\nu_q} \partial_{\mu}\psi^{\lambda}_{\bar{\nu_1}\cdots\bar{\nu_q}}(z)d\overline{z^{\nu_1}} \wedge \cdots \wedge d\overline{z^{\nu_q}}$.

Obsere that $[\varphi,\psi]$ is independent of the choice of local complex coordinates $(z^1,\dots,z^n)$ used for its definition and $[\varphi,\psi] \in \mathcal{L}^{0,p+q}(T)$. Moreover, we have the following formulae :

$$[\psi,\varphi] = -(-1)^{pq}[\varphi,\psi], \tag{2.31}$$

$$\bar{\partial}[\varphi,\psi] = [\bar{\partial}\varphi,\psi] + (-1)^{p}[\varphi,\bar{\partial}\psi], \tag{2.32}$$

$$(-1)^{pr}[[\varphi,\psi],\tau] + (-1)^{qp}[[\psi,\tau],\varphi] + (-1)^{rq}[[\tau,\varphi],\psi] = 0, \tag{2.33}$$

where $\varphi \in \mathcal{L}^{0,p}(T)$, $\psi \in \mathcal{L}^{0,q}(T)$ and $\tau \in \mathcal{L}^{0,r}(T)$.

Lemma 2.5.6 $\bar{\partial}\varphi(t) = \frac{1}{2}[\varphi(t),\varphi(t)]$, *where* $\bar{\partial}\varphi(t) = \sum_{\lambda} \bar{\partial}\varphi^{\lambda}\partial_{\lambda}$.

Proof : For simplicity, we denote $\zeta_j^\alpha(z,t)$ and $\varphi^\lambda(z,t)$ by ζ_j^α and φ^λ respectively. Applying $\bar\partial$ to (2.27) :

$$\bar\partial\zeta_j^\alpha = \sum_\lambda \varphi^\lambda \partial_\lambda \zeta_j^\alpha,$$

we get

$$0 = \sum_\lambda \bar\partial\varphi^\lambda \partial_\lambda \zeta_j^\alpha - \sum_\lambda \varphi^\lambda \wedge \bar\partial\partial_\lambda \zeta_j^\alpha.$$

Hence, we obtain

$$\sum_\lambda \bar\partial\varphi^\lambda \partial_\lambda \zeta_j^\alpha = \sum_\mu \varphi^\mu \wedge \bar\partial\partial_\mu \zeta_j^\alpha. \tag{2.34}$$

As $\bar\partial_\nu \zeta_j^\alpha = \sum\limits_\lambda \varphi_{\bar\nu}^\lambda \partial_\lambda \zeta_j^\alpha$, we have

$$\begin{aligned}
\bar\partial\partial_\mu\zeta_j^\alpha &= \sum_\nu \bar\partial_\nu\partial_\mu\zeta_j^\alpha d\bar z^\nu \\
&= \sum_\nu \partial_\mu\bar\partial_\nu\zeta_j^\alpha d\bar z^\nu \\
&= \sum_\nu \left(\partial_\mu\left(\sum_\lambda \varphi_{\bar\nu}^\lambda \partial_\lambda\zeta_j^\alpha\right)\right) d\bar z^\nu \\
&= \sum_\lambda \left(\sum_\nu \partial_\mu\varphi_{\bar\nu}^\lambda d\bar z^\nu\right)\partial_\lambda\zeta_j^\alpha + \sum_\lambda \varphi^\lambda\partial_\mu\partial_\lambda\zeta_j^\alpha.
\end{aligned}$$

Hence, we get

$$\begin{aligned}
\sum_\mu \varphi^\mu \wedge \bar\partial\partial_\mu\zeta_j^\alpha &= \sum_\lambda \left(\sum_\mu \varphi^\mu \wedge \partial_\mu\varphi^\lambda\right)\partial_\lambda\zeta_j^\alpha + \sum_{\lambda,\mu} \varphi^\mu \wedge \varphi^\lambda\partial_\mu\partial_\lambda\zeta_j^\alpha \\
&= \sum_\lambda \left(\sum_\mu \varphi^\mu \wedge \partial_\mu\varphi^\lambda\right)\partial_\lambda\zeta_j^\alpha,
\end{aligned}$$

where $\partial_\mu\varphi^\lambda = \sum\limits_\nu \partial_\mu\varphi_{\bar\nu}^\lambda d\bar z^\nu$. Therefore, (2.34) becomes

$$\sum_\lambda \bar\partial\varphi^\lambda\partial_\lambda\zeta_j^\alpha = \sum_\lambda \left(\sum_\mu \varphi^\mu \wedge \partial_\mu\varphi^\lambda\right)\partial_\lambda\zeta_j^\alpha.$$

As $\det\left(\partial_\lambda \zeta_j^\alpha\right) \neq 0$, we have

$$\bar{\partial}\varphi^\lambda = \sum_\mu \varphi^\mu \wedge \partial_\mu \varphi^\lambda.$$

Using the bracket notation, we get

$$\bar{\partial}\varphi(t) = \frac{1}{2}[\varphi(t), \varphi(t)]. \tag{2.35}$$

□

Theorem 2.5.7 $\left(\frac{\partial\varphi(t)}{\partial t}\right)_{t=0}$ *is a $\bar{\partial}-$closed $(0,1)-$form, and under the isomorphism*

$$H^1(M, \Theta) \cong \mathcal{L}_{\bar{\partial}}^{0,1}(T)/\bar{\partial}\mathcal{L}^{0,0}(T),$$

$\left(\frac{\partial M_t}{\partial t}\right)_{t=0} \in H^1(M,\Theta)$ *corresponds to* $-\left(\frac{\partial\varphi(t)}{\partial t}\right)_{t=0} \in \mathcal{L}_{\bar{\partial}}^{0,1}(T)$.

Proof : The exact sequence of sheaves :

$$0 \to \Theta \to \mathcal{A}(T) \to \bar{\partial}\mathcal{A}(T) \to 0$$

induces the exact sequence of cohomology groups :

$$0 \to H^0(M,\Theta) \to \mathcal{L}^{0,0}(T) \xrightarrow{\bar{\partial}} \mathcal{L}_{\bar{\partial}}^{0,1}(T) \xrightarrow{\delta^*} H^1(M,\Theta) \to 0.$$

In order to prove the theorem, by the definition of δ^*[10], it suffices to show that there exists a 0-cochain $\{\xi_j\} \in C^0(\mathcal{U}, \mathcal{A}(T))$ such that

$$\bar{\partial}\{\xi_j\} = \left(\frac{\partial\varphi(t)}{\partial t}\right)_{t=0} \quad \text{and} \quad \delta\{\xi_j\} = -\{\theta_{jk}\}.$$

For simplicity, we denote $\left(\frac{\partial f(t)}{\partial t}\right)_{t=0}$ by $\dot{f}$ for any C^∞ function of $t \in \Delta$. Differentiating the fundamental equalities

$$\zeta_j^\alpha(z,t) = f_{jk}^\alpha(\zeta_k(z,t),\ t)$$

with respect to t, and putting $t=0$, we get

$$\dot{\zeta}_j^\alpha = \sum_\beta \frac{\partial \zeta_j^\alpha}{\partial \zeta_k^\beta}\dot{\zeta}_k^\beta + \dot{f}_{jk}^\alpha.$$

Putting $\xi_j = \sum_\alpha \dot{\zeta}_j^\alpha \frac{\partial}{\partial \zeta_j^\alpha}$, we have

$$\xi_j = \xi_k + \theta_{jk}.$$

That is

$$\delta\{\xi_j\} = -\{\theta_{jk}\}.$$

Moreover, differentiating

$$\bar{\partial}\zeta_j^\alpha(z,t) = \sum_\lambda \varphi^\lambda(z,t)\partial_\lambda \zeta_j^\alpha(z,t)$$

with respect to t, and putting $t=0$, we get

$$\bar{\partial}\dot{\zeta}_j^\alpha = \sum_\lambda \dot{\varphi}^\lambda \partial_\lambda \zeta_j^\alpha$$

since $\varphi^\lambda(z,0)=0$. Therefore, we have

$$\bar{\partial}\xi_j = \sum_\alpha \bar{\partial}\dot{\zeta}_j^\alpha \frac{\partial}{\partial \zeta_j^\alpha} = \sum_\alpha \sum_\lambda \dot{\varphi}^\lambda \frac{\partial \zeta_j^\alpha}{\partial z^\lambda}\frac{\partial}{\partial \zeta_j^\alpha} = \sum_\lambda \dot{\varphi}^\lambda \partial_\lambda = \dot{\varphi}.$$

□

Now, we want to look at the meaning of (2.35) :

$$\bar{\partial}\varphi(t) = [\varphi(t), \varphi(t)],$$

where $\varphi(t) = \sum_\lambda \varphi^\lambda(z,t)\partial_\lambda$ and $\varphi^\lambda(z,t) = \sum_\nu \varphi_{\bar{\nu}}^\lambda(z,t)d\bar{z^\nu}$. We fix t for a moment and consider $\varphi_{\bar{\nu}}^\lambda$ as a C^∞ funtion on $U \subset M$ with local coordinates $(z^1, \ldots, z^n)$. Note that

$$(\bar{\partial} - \sum_\lambda \varphi^\lambda \partial_\lambda)f = 0$$

is equivalent to the following system of partial differential equations :

$$L_\nu f = 0, \text{ where } L_\nu = \bar{\partial}_\nu - \sum_\lambda \varphi_{\bar{\nu}}^\lambda \partial_\lambda, \ \nu = 1, \dots, n. \tag{2.36}$$

Lemma 2.5.8 *The differential operators $L_1, \dots, L_n, \bar{L_1}, \dots, \bar{L_n}$ are linearly independent.*

Proof : Suppose that

$$\sum_\lambda a_\lambda(\bar{\partial}_\lambda - \sum_\mu \varphi_{\bar{\lambda}}^\mu \partial_\mu) + \sum_\mu b_\mu(\partial_\mu - \sum_\lambda \overline{\varphi_{\bar{\mu}}^\lambda} \bar{\partial}_\lambda) = 0,$$

then we have

$$a_\lambda = \sum_\mu \overline{\varphi_{\bar{\mu}}^\lambda} b_\mu \quad \text{and} \quad b_\mu = \sum_\nu \varphi_{\bar{\nu}}^\mu a_\nu.$$

Hence, we get

$$a_\lambda = \sum_{\mu,\nu} \varphi_{\bar{\nu}}^\mu \overline{\varphi_{\bar{\mu}}^\lambda} a_\nu.$$

Since $\det\left(\delta_\nu^\lambda - \sum_\mu \varphi_{\bar{\nu}}^\mu \overline{\varphi_{\bar{\mu}}^\lambda}\right)$ is assumed to be nonzero, we have $a_\lambda = 0$ for $\lambda = 1, \dots, n$, hence also $b_\mu = 0$ for $\mu = 1, \dots, n$. □

Observe that (2.35) is written in the explicit form as

$$\bar{\partial}_\tau \varphi_{\bar{\nu}}^\lambda - \bar{\partial}_\nu \varphi_{\bar{\tau}}^\lambda = \sum_\mu \varphi_{\bar{\tau}}^\mu \partial_\mu \varphi_{\bar{\nu}}^\lambda - \sum_\mu \varphi_{\bar{\nu}}^\mu \partial_\mu \varphi_{\bar{\tau}}^\lambda,$$

and is equivalent to the system of equations

$$L_\tau L_\nu - L_\nu L_\tau = 0, \quad \tau, \nu = 1, \dots, n. \tag{2.37}$$

We see from the proof of Lemma(2.5.6) that (2.37) is a necessary condition for (2.36) to have n C^∞ solutions

$$f = \zeta_j^\alpha = \zeta_j^\alpha(z), \ \alpha = 1, \dots, n, \quad \text{with} \quad \det\left(\partial_\lambda \zeta_j^\alpha\right) \neq 0.$$

For the sufficiency of (2.37), we use Newlander-Nirenberg Theorem [12]:

Theorem 2.5.9 *(Newlander-Nirenberg)[12] Let* $\varphi = \sum_{\lambda,\nu} \varphi^\lambda_{\bar\nu} d\bar{z}^\nu \partial_\lambda$ *be a* C^∞ *vector* $(0,1)-$*form defined on a domain* U *of* $\mathbb{C}^n$, *and put* $L_\nu = \bar\partial_\nu - \sum_\lambda \varphi^\lambda_{\bar\nu}\partial_\lambda$. *Suppose that* $L_1, \ldots, L_n, \bar{L_1}, \ldots, \bar{L_n}$ *are linearly independent, and that they satisfy the condition (2.37) :*

$$L_\tau L_\nu - L_\nu L_\tau = 0, \quad \tau, \nu = 1, \ldots, n.$$

Then the system of partial differential equations (2.36)

$$L_\nu f = 0, \quad \nu = 1, \ldots, n.$$

has n *linearly independent* C^∞ *solutions*

$$f = \zeta^\alpha = \zeta^\alpha(z), \quad \alpha = 1, \ldots, n.$$

in a sufficiently small neighborhood of any point of U.

We see from the proof of Lemma(2.5.8) that $\det\left(\delta^\lambda_\nu - \sum_\mu \varphi^\mu_{\bar\nu}\overline{\varphi^\lambda_{\bar\mu}}\right) \neq 0$ implies $L_1, \ldots, L_n, \bar{L_1}, \ldots, \bar{L_n}$ are linearly independent. Consequently when $\det\left(\delta^\lambda_\nu - \sum_\mu \varphi^\mu_{\bar\nu}\overline{\varphi^\lambda_{\bar\mu}}\right) \neq 0$, (2.37) gives a necessary and sufficient condition for (2.36) to have n linearly independent C^∞ solutions

$$f = \zeta^\alpha = \zeta^\alpha(z), \quad \alpha = 1, \ldots, n,$$

in a neighborhood of any point of U. (2.37) is called the integrability condition for (2.36). Thus for $\varphi(t)$ on M, (2.35) :

$$\bar\partial\varphi(t) = \frac{1}{2}[\varphi(t), \varphi(t)]$$

is the integrability condition for (2.30) :

$$(\bar\partial - \varphi(t))f = 0.$$

Now, we summarize our results as the following theorem :

Theorem 2.5.10 *Let $\varphi = \sum_{\lambda} \varphi^{\lambda}\partial_{\lambda}$ be a C^{∞} vector $(0,1)-$form on a compact complex manifold M, where $\varphi^{\lambda} = \sum_{\nu} \varphi^{\lambda}_{\bar{\nu}} d\bar{z}^{\nu}$. Suppose that* $\det\left(\delta^{\lambda}_{\nu} - \sum_{\mu} \varphi^{\mu}_{\bar{\nu}} \overline{\varphi^{\lambda}_{\bar{\mu}}}\right)$ *$\neq 0$ and $\bar{\partial}\varphi = \frac{1}{2}[\varphi, \varphi]$. Then φ determines a complex structure M_{φ} on the compact differentiable manifold M.*

Proof : Let $\mathcal{U} = \{U_j\}$ be a sufficiently fine finite open covering of M. Then by Newlander-Nirenberg Theorem2.5.9, there are n linearly independent C^{∞} functions

$$\zeta^{\alpha}_j = \zeta^{\alpha}_j(z), \qquad \alpha = 1, \ldots, n,$$

on each U_j such that

$$\left(\bar{\partial} - \sum_{\lambda} \varphi^{\lambda}\partial_{\lambda}\right)\zeta^{\alpha}_j = 0.$$

Moreover, by Theorem(2.5.3), we have coordinate transformations f^{α}_{jk} on M:

$$\zeta^{\alpha}_j(z) = f^{\alpha}_{jk}(\zeta_k(z)) \text{ on } U_j \cap U_k \neq \emptyset, \qquad \alpha = 1, \ldots, n,$$

where f^{α}_{jk} are holomorphic in $\zeta^1_k(z), \ldots, \zeta^n_k(z)$. □

Observe that if φ is equal to $\varphi(t)$ defined by (2.27), then we have $M_{\varphi(t)} = M_t = \hat{\omega}^{-1}(t)$. With these preparations, we prove the theorem of existence by the following steps :

<u>**STEP 1**</u> :

Construct a family $\{\varphi(t) : t \in \Delta\}$ of C^{∞} vector $(0,1)-$forms $\varphi(t)$ on M with $0 \in \Delta \subset \mathbb{C}^m$ satisfying the following conditions :

(i) $\varphi(0) = 0$,

(ii) $\bar{\partial}\varphi(t) = \frac{1}{2}[\varphi(t), \varphi(t)]$,

(iii) $\left\{\left(\frac{\partial\varphi(t)}{\partial t_1}\right)_{t=0}, \ldots, \left(\frac{\partial\varphi(t)}{\partial t_m}\right)_{t=0}\right\}$ form a basis for $H^1(M, \Theta)$.

Note that **(i)** and **(ii)** are the conditions of Theorem(2.5.10) provided that Δ is sufficiently small.

<u>**STEP 2**</u> :

Prove that the family $\{M_{\varphi(t)}\}$ thus obtained is a complex analytic family.

<u>**STEP 1(a)**</u> : <u>**Construction of Power Series**</u>

We first introduce some notation. For power series $A(t) = A(t_1, \ldots, t_m)$ and $B(t) = B(t_1, \ldots, t_m)$ in $t_1, \ldots, t_m$, we put

$$A(t)_\nu = A_\nu(t) = \sum_{\nu_1+\cdots+\nu_m=\nu} A_{\nu_1\cdots\nu_m} t_1^{\nu_1} \cdots t_m^{\nu_m} \text{ and } A^\nu(t) = A_0(t) + \cdots + A_\nu(t).$$

Then

$$A(t) = \sum_{\nu=0}^{\infty} A_\nu(t) \quad \text{and} \quad \text{by } A(t) \overset{\nu}{\equiv} B(t) \text{ we mean } A^\nu(t) = B^\nu(t).$$

We take an arbitrary basis $\{\beta_1, \ldots, \beta_m\}$ for $H^1(M, \Theta)$ and put

$$\varphi_0 = 0, \ \varphi_1(t) = \sum_\lambda \beta_\lambda t_\lambda.$$

Then it is clear that

$$\varphi(t) = \varphi_1(t) + \varphi_2(t) + \varphi_3(t) + \cdots$$

satisfies the conditions **(i)** and **(iii)**. Hence it remains to determine $\varphi_2(t), \cdots, \varphi_\nu(t), \cdots$ such that $\varphi(t) = \varphi(z,t)$ is C^∞ on M for every $t \in \Delta$ and condition **(ii)** holds.

Since $\beta_\lambda \in \mathcal{L}^{0,1}_{\bar\partial}(T)$, we have $\bar\partial\varphi_1(t) = 0$. Moreover, as $\varphi_0(t) = 0$ and $[\varphi_\mu(t), \varphi_\nu(t)]$ is a homogeneous polynomial of degree $\mu + \nu$ in $t_1, \cdots, t_m$, we have

$$[\varphi^\nu(t), \varphi^\nu(t)] \overset{\nu}{\equiv} [\varphi^{\nu-1}(t), \varphi^{\nu-1}(t)].$$

Consequently, the condition **(ii)** is reduced to the system of infinitely many congruences :

$$(\mathbf{J}_\nu)\cdots\cdots\ \bar{\partial}\varphi^\nu(t) \overset{\nu}{\equiv} \frac{1}{2}[\varphi^{\nu-1}(t), \varphi^{\nu-1}(t)], \qquad \nu = 2, 3, \cdots$$

Thus in order to obtain the required $\varphi(t)$, it suffices to determines $\varphi_\nu(t)$ satisfying $(\mathbf{J}_\nu)$ by induction on ν. For $\nu = 2$, we put

$$\psi_2(t) = \frac{1}{2}[\varphi^1(t), \varphi^1(t)] = \frac{1}{2}[\varphi_1(t), \varphi_1(t)].$$

By (2.31) and (2.32), we have

$$\bar{\partial}\psi_2(t) = [\bar{\partial}\varphi_1(t), \varphi_1(t)] = 0.$$

Therefore, we obtain

$$\psi_2 = \sum_{\lambda,\mu} \psi_{\lambda\mu} t_\lambda t_\mu \quad \text{with} \quad \psi_{\lambda\mu} = \psi_{\mu\lambda} \in \mathcal{L}^{0,2}_{\bar{\partial}}(T).$$

As $H^2(M, \Theta) = 0$, by Dolbeault's Theorem, we have

$$H^2(M, \Theta) = \mathcal{L}^{0,2}_{\bar{\partial}}(T)/\bar{\partial}\mathcal{L}^{0,1}(T) = 0.$$

Hence, there is a $\varphi_{\lambda\mu} \in \mathcal{L}^{0,1}(T)$ such that

$$\bar{\partial}\varphi_{\lambda\mu} = \psi_{\lambda\mu}.$$

Thus, putting $\varphi_2(t) = \sum_{\lambda,\mu} \varphi_{\lambda\mu} t_\lambda t_\mu$, we get

$$\bar{\partial}\varphi_2(t) = \psi_2(t) = \frac{1}{2}[\varphi_1(t), \varphi_1(t)].$$

Therefore, $(\mathbf{J_2})$ holds. Assume that

$\varphi^\nu(t) = \varphi_1(t) + \cdots + \varphi_\nu(t)$ is already determined and $(\mathbf{J}_\nu)$ holds.

Then we consider $(\mathbf{J}_{\nu+1})$:

$$\bar{\partial}\varphi^{\nu}(t)+\bar{\partial}\varphi_{\nu+1}(t)\overset{\nu+1}{\equiv}\frac{1}{2}[\varphi^{\nu}(t),\varphi^{\nu}(t)].$$

By assumption, we have

$$\bar{\partial}\varphi^{\nu}(t)\overset{\nu}{\equiv}\frac{1}{2}[\varphi^{\nu-1}(t),\varphi^{\nu-1}(t)]\overset{\nu}{\equiv}\frac{1}{2}[\varphi^{\nu}(t),\varphi^{\nu}(t)].$$

Let

$$\psi_{\nu+1}(t)=\left(\frac{1}{2}[\varphi^{\nu}(t),\varphi^{\nu}(t)]\right)_{\nu+1}.$$

Then in order to prove $(\mathbf{J}_{\nu+1})$, it suffices to show that there exists $\varphi_{\nu+1}(t)$ such that

$$\bar{\partial}\varphi_{\nu+1}(t)=\psi_{\nu+1}(t).$$

Note that

$$\psi_{\nu+1}(t)\overset{\nu}{\equiv}\frac{1}{2}[\varphi^{\nu}(t),\varphi^{\nu}(t)].$$

By (2.31) and (2.32), we have

$$\bar{\partial}\psi_{\nu+1}(t)\overset{\nu}{\equiv}[\bar{\partial}\varphi^{\nu}(t),\varphi^{\nu}(t)].$$

As $\bar{\partial}\varphi^{\nu}(t)\overset{\nu}{\equiv}\frac{1}{2}[\varphi^{\nu}(t),\varphi^{\nu}(t)]$ and $\varphi^{0}(t)=0$, we have

$$\bar{\partial}\psi_{\nu+1}(t)\overset{\nu+1}{\equiv}\frac{1}{2}[[\varphi^{\nu}(t),\varphi^{\nu}(t)],\ \varphi^{\nu}(t)].$$

Since $[[\varphi^{\nu}(t),\varphi^{\nu}(t)],\ \varphi^{\nu}(t)]=0$ by (2.33), we get

$$\bar{\partial}\psi_{\nu+1}(t)=0.$$

Since $H^2(M,\Theta)=0$, by the same argument as before, there exists $\varphi_{\nu+1}(t)=\sum_{\nu_1+\cdots+\nu_m=\nu+1}\varphi_{\nu_1\cdots\nu_m}t_1^{\nu_1}\cdots t_m^{\nu_m}$ with $\varphi_{\nu_1\cdots\nu_m}\in\mathcal{L}^{0,1}(T)$ such that

$$\bar{\partial}\varphi_{\nu+1}(t)=\psi_{\nu+1}(t).$$

Thus we have constructed a power series

$$\varphi(t) = \varphi_0(t) + \varphi_1(t) + \varphi_2(t) + \cdots$$

satisfying the conditions **(i)**, **(ii)** and **(iii)**. Next, we must prove that the power series $\varphi(t)$ thus obtained converges for $t \in \Delta$ provided that Δ is sufficiently small.

STEP 1(b) : **Proof of Convergence**

We introduce a Hermitian metric

$$\sum_{\lambda,\mu=1}^{n} g_{\lambda\bar{\nu}} dz^\lambda \otimes d\bar{z^\nu} \quad \text{on} \quad M,$$

and define the dual forms, inner products, norms, etc. of differential forms on M as usual. For the tangent bundle T, we define its Hermitian metric on fibres by

$$\sum_{\lambda,\nu=1}^{n} g_{\lambda\bar{\nu}} \zeta^\lambda \bar{\zeta^\nu} \quad \text{for} \quad \sum_{\lambda=1}^{n} \zeta^\lambda \partial_\lambda \in T.$$

Then the inner product of vector $(0,q)-$forms $\varphi = \sum_\lambda \varphi^\lambda \partial_\lambda$ and $\psi = \sum_\lambda \psi^\lambda \partial_\lambda$ on M is given by(For details, see [10])

$$(\varphi, \psi) = \int_M \sum_{\lambda,\nu} g_{\lambda\bar{\nu}} \varphi^\lambda \wedge *\bar{\psi^\nu},$$

where $*\bar{\psi^\nu}$ is the dual form of $\bar{\psi^\nu}$. Let ϑ be the adjoint operator of $\bar{\partial}$ with respect to this inner product. Then we have

$$(\bar{\partial}\varphi, \psi) = (\varphi, \vartheta\psi), \ \varphi \in \mathcal{L}^{0,q}(T), \ \psi \in \mathcal{L}^{0,q+1}(T).$$

Moreover, the complex Laplace-Beltram operator $\Box$ is given by

$$\Box = \bar{\partial}\vartheta + \vartheta\bar{\partial} = -\bar{\partial} * \partial * - * \partial * \bar{\partial}.$$

Definition 2.5.11 *$\varphi \in \mathcal{L}^{p,q}(T)$ is called a harmonic vector $(p,q)-$form if $\bar{\partial}\varphi = \vartheta\varphi = 0$.*

Note that $\varphi \in \mathcal{L}^{p,q}(T)$ is harmonic if and only if $\Box\varphi = 0$. Put

$$\mathbb{H}^{p,q}(T) = \{\varphi \in \mathcal{L}^{p,q}(T) \ : \ \Box\varphi = 0\}.$$

Then, the following formulae are fundamental :

$$\mathcal{L}^{0,q}(T) = \mathbb{H}^{0,q}(T) \oplus \Box\mathcal{L}^{0,q}(T), \tag{2.38}$$

$$\mathcal{L}^{0,q}_{\bar{\partial}}(T) = \mathbb{H}^{0,q}(T) \oplus \bar{\partial}\mathcal{L}^{0,q-1}(T), \tag{2.39}$$

$$H^q(M,\Theta) \cong \mathcal{L}^{0,q}_{\bar{\partial}}/\bar{\partial}\mathcal{L}^{0,q-1}(T) \cong \mathbb{H}^{0,q}(T). \tag{2.40}$$

Definition 2.5.12 *Let $f(x) = f(x^1, \cdots, x^{2n})$ be a complex-valued C^∞ function defined on a domain U of $\mathbb{R}^n$. Then for $k \geq 0$ and $0 < \alpha < 1$, the Hölder norm $|f|^U_{k+\alpha}$ of f is defined by*

$$|f|^U_{k+\alpha} = \sum_{h=0}^{k}\sum_{D^h}\sup_{x\in U}|D^h f(x)| + \sum_{D^k}\sup_{x,y\in U}\frac{|D^k f(x) - D^k f(y)|}{|x-y|^\alpha},$$

where $D^h = \left(\frac{\partial}{\partial x^1}\right)^{h_1}\cdots\left(\frac{\partial}{\partial x^{2n}}\right)^{h_{2n}}$, $h_1+\cdots+h_{2n} = h$ and $|x-y| = \sqrt{\sum_\nu |x^\nu - y^\nu|^2}$.

For a C^∞ vector $(0,q)-$form $\varphi \in \mathcal{L}^{0,q}(T)$ on M, we define its Hölder norm as follows :

Let $\{U_j\}$ be a finite open covering of M, where U_j is a coordinate polydisk. We choose local complex coordinates $\{z_j\}$ with $z_j = (z_j^1, \ldots, z_j^n)$ such that the closure $\overline{U_j}$ of U_j is contained in the domain of z_j and

$$U_j = \{z_j \ : \ |z_j^1| < 1, \ldots, |z_j^n| < 1\}.$$

In terms of local coordinates, $\varphi \in \mathcal{L}^{0,q}(T)$ is represented as

$$\varphi = \sum_{\lambda} \frac{1}{q!} \varphi^{\lambda}_{j\bar{\nu}_1 \cdots \bar{\nu}_q}(z_j) d\bar{z}_j^{\nu_1} \cdots d\bar{z}_j^{\nu_q} \frac{\partial}{\partial z_j^{\lambda}} \quad \text{on each} \quad U_j.$$

Putting $z_j^{\nu} = x_j^{2\nu-1} + i x_j^{2\nu}$, we get the real coordinates $x_j = (x_j^1, \ldots, x_j^{2n})$. Then $\varphi^{\lambda}_{j\bar{\nu}_1 \cdots \bar{\nu}_q} = \varphi^{\lambda}_{j\bar{\nu}_1 \cdots \bar{\nu}_q}(x_j)$ can be considered as C^{∞} functions on $U_j \subset \mathbb{R}^{2n}$. Put

$$|\varphi|_{k+\alpha} = \max_{j} \max_{\lambda, \nu_1, \cdots, \nu_q} |\varphi^{\lambda}_{j\bar{\nu}_1 \cdots \bar{\nu}_q}|^{U_j}_{k+\alpha}$$

and

$$|\varphi|_0 = \max_{j} \max_{\lambda, \nu_1, \cdots, \nu_q} \sup_{x_j \in U_j} |\varphi^{\lambda}_{j\bar{\nu}_1 \cdots \bar{\nu}_q}(x_j)|.$$

As $|\varphi|_{k+\alpha}$ depends on the choice of $\{U_j\}$ and $\{z_j\}$, we fix $\{U_j\}$ and $\{z_j\}$ once for all below. Now, we state a fundamental inequality without proof :

Theorem 2.5.13 *[2]For $k \geq 2$, the inequality*

$$|\varphi|_{k+\alpha} \leq c(|\Box\varphi|_{k-2+\alpha} + |\varphi|_0) \tag{2.41}$$

holds for $\varphi \in \mathcal{L}^{0,q}(T)$, where c is a constant depending only on k and α.

Lemma 2.5.14 *If $H^2(M, \Theta) = 0$, then the linear map*

$$\Box \ : \ \mathcal{L}^{0,2}(T) \to \mathcal{L}^{0,2}(T) \ \textit{is bijective.}$$

Proof : The result follows directly froms (2.38) and (2.40). $\Box$

We denote the inverse of $\Box$ by $G \ : \ G = \Box^{-1}$. G is called the Green operator.

Lemma 2.5.15 *For any $\psi \in \mathcal{L}^{0,2}_{\bar{\partial}}(T)$, we have*

$$\psi = \bar{\partial}\vartheta G\psi. \tag{2.42}$$

Proof : Applying $\bar{\partial}$ to the equality

$$\psi = \Box G\psi = \bar{\partial}\vartheta G\psi + \vartheta\bar{\partial}G\psi,$$

we get

$$0 = \bar{\partial}\psi = \partial\vartheta\bar{\partial}G\psi.$$

Therefore, we have

$$\|\vartheta\bar{\partial}G\psi\|^2 = (\vartheta\bar{\partial}G\psi, \vartheta\bar{\partial}G\psi) = (\bar{\partial}\vartheta\bar{\partial}G\psi, \bar{\partial}G\psi) = 0.$$

This implies that $\vartheta\bar{\partial}G\psi = 0$ and thus $\psi = \bar{\partial}\vartheta G\psi$. $\Box$

Lemma 2.5.16 *For $k \geq 2$, the inequality*

$$|G\psi|_{k+\alpha} \leq c_1|\psi|_{k-2+\alpha} \tag{2.43}$$

holds for any $\psi \in \mathcal{L}^{0,2}(T)$, where c_1 is a constant independent of ψ.

Proof : By Theorem2.5.13, we have

$$|G\psi|_{k+\alpha} \leq c(|\psi|_{k-2+\alpha} + |G\psi|_0) \quad \text{for any} \quad \psi \in \mathcal{L}^{0,2}(T). \tag{2.44}$$

Thus in order to prove (2.43), it suffices to show that

$$|G\psi|_0 \leq c_2|\psi|_{k-2+\alpha} \quad \text{for any} \quad \psi \in \mathcal{L}^{0,2}(T), \tag{2.45}$$

where c_2 is a constant independent of ψ.

Suppose that (2.45) is not true for any c_2. Then for any positive integer n, there exists $\psi^{(n)} \in \mathcal{L}^{0,2}(T)$ such that

$$|G\psi^{(n)}|_0 > n|\psi^{(n)}|_{k-2+\alpha}. \tag{2.46}$$

Clearly, we may assume that $|G\psi^{(n)}|_0 = 1$, then (2.46) becomes

$$|G\psi^{(n)}|_0 = 1 \text{ and } |\psi^{(n)}|_{k-2+\alpha} < \frac{1}{n}. \tag{2.47}$$

Putting $\varphi^{(n)} = G\psi^{(n)}$, by (2.44), we have

$$|\varphi^{(n)}|_{k+\alpha} < 2c.$$

Representing $\varphi^{(n)}$ on each U_j in the form

$$\varphi^{(n)} = \sum_{\lambda} \frac{1}{2} \sum_{\beta,\gamma} \varphi^{(n)\lambda}_{j\beta\gamma}(x_j) d\bar{z}^{\beta}_j \wedge d\bar{z}^{\gamma}_j \frac{\partial}{\partial z^{\lambda}_j},$$

we obtain

$$|\varphi^{(n)\lambda}_{j\beta\gamma}|^{U_j}_{k+\alpha} < 2c.$$

We first fix $j, \lambda, \beta, \gamma$ arbitrarily, and put

$$f_n = f_n(x_j) = \varphi^{(n)\lambda}_{j\beta\gamma}(x_j).$$

Since $|f_n|^{U_j}_{k+\alpha} < 2c$, by Definition2.5.12, we obtain the following facts on U_j :
(i) $|D^h f_n(x_j)| < 2c$ and
(ii) the sequence of functions $\{D^h f_n(x_j)\}$ is equicontinuous for each $h \leq k$.
Consequently, by Ascoli's Theorem, we may assume that $\{D^h f_n(x_j)\}$ converges uniformly on U_j for each $h \leq k$. Put

$$f(x_j) = \lim_{n\to\infty} f_n(x_j).$$

Then $f(x_j)$ is a C^k function on U_j, and $\{D^h f_n(x_j)\}$ converges uniformly to $D^h f(x_j)$ for each $h \leq k$. Since the indices $j, \lambda, \beta, \gamma$ are arbitrarily fixed, we may assume that there is a C^k vector $(0,2)$−form φ such that the sequence

$\{D^h\varphi_{j\beta\gamma}^{(n)\lambda}(x_j)\}$ converges uniformly to $D^h\varphi_{j\beta\gamma}^{\lambda}(x_j)$ on U_j for each $h \leq k$. Therefore, $\varphi^{(n)}$ converges uniformly to φ and $\Box\varphi^{(n)}$ converges uniformly to $\Box\varphi$ since $k \geq 2$. Since

$$\varphi^{(n)} = G\psi^{(n)} \quad \text{and} \quad \Box\varphi^{(n)} = \Box G\psi^{(n)} = \psi^{(n)},$$

we have

$$|G\psi^{(n)} - \varphi|_0 \to 0 \quad \text{as} \quad n \to \infty \tag{2.48}$$

and

$$|\psi^{(n)} - \Box\varphi|_0 \to 0 \quad \text{as} \quad n \to \infty. \tag{2.49}$$

On the other hand, by (2.47), we have

$$|\psi^{(n)}|_0 \leq |\psi^{(n)}|_{k-2+\alpha} \to 0 \quad \text{as} \quad n \to \infty.$$

Consequently, we have $\Box\varphi = 0$ and hence

$$(\varphi, \psi) = (\varphi, \Box G\psi) = (\Box\varphi, G\psi) = 0 \quad \text{for any} \quad \psi \in \mathcal{L}^{0,2}(T).$$

This implies that $\varphi = 0$. Therefore, by (2.48), we get

$$|G\psi^{(n)}|_0 \to 0 \quad \text{as} \quad n \to \infty,$$

which contradicts the fact that $|G\psi^{(n)}|_0 = 1$ for all n. $\Box$

Definition 2.5.17 *Let*

$$P(t) = P(t_1, \ldots, t_m) = \sum_{\nu_1,\ldots,\nu_m} P_{\nu_1\cdots\nu_m} t_1^{\nu_1} \cdots t_m^{\nu_m}$$

be a power series in $t_1, \cdots, t_m$. *A power series*

$$a(t) = \sum_{\nu_1,\ldots,\nu_m} a_{\nu_1\cdots\nu_m} t_1^{\nu_1} \cdots t_m^{\nu_m} \quad \text{with} \quad a_{\nu_1\cdots\nu_m} \geq 0$$

is said to be a majorant of $P(t)$ *if*

$$|P_{\nu_1\cdots\nu_m}| \leq a_{\nu_1\cdots\nu_m}, \qquad \nu_1, \ldots, \nu_m = 0, 1, 2, \ldots$$

and is denoted by $P(t) << a(t)$.

Definition 2.5.18 *For a power series*

$$\psi(t) = \sum_{\nu_1,\ldots,\nu_m} \psi_{\nu_1\cdots\nu_m} t_1^{\nu_1} \cdots t_m^{\nu^m} \quad \text{with} \quad \psi_{\nu_1\cdots\nu_m} \in \mathcal{L}^{0,q}(T),$$

we define $|\psi|_{k+\alpha}(t)$ *by*

$$|\psi|_{k+\alpha}(t) = \sum_{\nu_1,\ldots,\nu_m} |\psi_{\nu_1\cdots\nu_m}|_{k+\alpha} t_1^{\nu_1} \cdots t_m^{\nu_m},$$

and write $|\psi|_{k+\alpha}(t) << a(t)$ *if*

$$|\psi_{\nu_1\cdots\nu_m}|_{k+\alpha} \leq a_{\nu_1\cdots\nu_m} \text{ with } a_{\nu_1\cdots\nu_m} \geq 0, \quad \nu_1, \ldots, \nu_m = 0, 1, 2, \ldots$$

With these preparations, we prove the convergence of $\varphi(t)$ as follows :

Observe that in order to prove the convergence of the power series $\varphi(t) = \sum_{\nu} \varphi_\nu(t)$ on some polydisk $\Delta_\varepsilon = \{t \in \mathbb{C}^m \ : \ |t_1| < \varepsilon, \ldots, |t_m| < \varepsilon, \ \varepsilon > 0\}$ with respect to the Hölder norm, it suffices to show that there is a power series $a(t)$ which converges absolutely in Δ_ε such that

$$|\varphi|_{k+\alpha}(t) << a(t).$$

As stated in **STEP 1(a)**, we have

$$\psi_{\nu+1}(t) = \left(\frac{1}{2}[\varphi^\nu(t), \varphi^\nu(t)]\right)_{\nu+1} \qquad \text{and} \qquad \bar\partial\psi_{\nu+1}(t) = 0.$$

Therefore, if we put

$$\varphi_{\nu+1}(t) = \vartheta G \psi_{\nu+1}(t),$$

it follows from Lemma 2.5.15 that $\varphi_{\nu+1}(t)$ is a solution of the equation

$$\bar\partial\varphi_{\nu+1}(t) = \psi_{\nu+1}(t).$$

Consequently, if we put

$$\varphi^1(t) = \varphi_1(t) = \sum_\lambda \beta_\lambda t_\lambda$$

and define $\varphi_{\nu+1}(t)$ successively for $\nu = 1, 2, 3, \cdots$ by

$$\varphi_{\nu+1}(t) = \vartheta G \psi_{\nu+1}(t), \tag{2.50}$$

then we obtain the power series $\varphi(t) = \sum_\nu \varphi_\nu(t)$ satisfying the conditions **(i)**, **(ii)** and **(iii)**.

Lemma 2.5.19 *Let* $A(t) = \frac{b}{16c}\sum_{\nu=1}^{\infty}\frac{c^\nu(t_1+\cdots+t_m)^\nu}{\nu^2}$ *with* $b, c > 0$. *Then, we have the following inequality :*

$$A(t)^2 << \frac{b}{c}A(t) \tag{2.51}$$

Proof : For simplicity, we consider the power series

$$B(s) = \sum_{\nu=1}^{\infty}\frac{s^\nu}{\nu^2} \qquad \text{in } s.$$

Then

$$B(s)^2 = \left(\sum_\lambda \frac{s^\lambda}{\lambda^2}\right)\left(\sum_\mu \frac{s^\mu}{\mu^2}\right) = \sum_{\nu=2}^{\infty} s^\nu \sum_{\lambda+\mu=\nu} \frac{1}{\lambda^2\mu^2}.$$

Note that

$$\sum_{\lambda+\mu=\nu} \frac{1}{\lambda^2\mu^2} \leq 2 \sum_{\substack{\lambda+\mu=\nu \\ \lambda\leq\mu}} \frac{1}{\lambda^2\mu^2} < 2\sum_{\lambda=1}^{\infty} \frac{4}{\lambda^2\nu^2} = \frac{8}{\nu^2}\sum_{\lambda=1}^{\infty}\frac{1}{\lambda^2} = \frac{8}{\nu^2}\frac{\pi^2}{6} < \frac{16}{\nu^2}.$$

Therefore we have

$$B(s)^2 << 16B(s).$$

Since $A(t) = \frac{b}{16c}B(c(t_1 + \cdots + t_m))$, we have

$$A(t)^2 << \frac{b}{c}A(t).$$

□

Theorem 2.5.20 *There exists positive numbers b and c such that*

$$|\varphi|_{k+\alpha}(t) << A(t) \quad \textit{for} \quad k \geq 2 \quad \textit{and} \quad 0 < \alpha < 1.$$

Proof : It suffices to show by induction that there exists positive numbers b and c such that

$$(\mathbf{K}_\nu) \cdots\cdots |\varphi^\nu|_{k+\alpha}(t) << A(t) \qquad \text{holds for all } \nu.$$

For $\nu = 1$, we have

$$\varphi^1(t) = \beta_1 t_1 + \cdots + \beta_m t_m.$$

Therefore, if we choose b such that

$$|\beta_\lambda|_{k+\alpha} < \frac{b}{16} \quad \text{for} \quad \lambda = 1, \ldots, m,$$

then $(\mathbf{K_1})$ holds. Now suppose that $(\mathbf{K}_\nu)$ holds and consider $(\mathbf{K}_{\nu+1})$. By the definition of Hölder norm, we have the following inequalities :

$$|\vartheta\psi|_{k+\alpha} \leq K_1|\psi|_{k+1+\alpha}, \quad \psi \in \mathcal{L}^{0,2}(T), \tag{2.52}$$

$$|[\varphi,\psi]|_{k-1+\alpha} \leq K_2|\varphi|_{k+\alpha}|\psi|_{k+\alpha}, \quad \varphi,\ \psi \in \mathcal{L}^{0,1}(T), \tag{2.53}$$

where $K_1,\ K_2$ are constants independent of φ and ψ. By (2.52) and (2.43), we have

$$|\varphi_{\nu+1}|_{k+\alpha}(t) = |\vartheta G\psi_{\nu+1}|_{k+\alpha}(t) \ll K_1|G\psi_{\nu+1}|_{k+1+\alpha}(t) \ll K_1c_1|\psi_{\nu+1}|_{k-1+\alpha}(t).$$

Since

$$\psi_{\nu+1}(t) = \left(\frac{1}{2}[\varphi^\nu(t),\varphi^\nu(t)]\right)_{\nu+1},$$

by (2.53), we have

$$|\varphi_{\nu+1}|_{k+\alpha}(t) << K_1c_1K_2|\varphi^\nu|_{k+\alpha}(t)|\varphi^\nu|_{k+\alpha}(t).$$

By the induction hypothesis and (2.51), we get

$$|\varphi_{\nu+1}|_{k+\alpha}(t) << K_1K_2c_1A(t)^2 << \frac{K_1K_2c_1b}{c}A(t).$$

Hence by putting $c = K_1K_2c_1b$, we obtain

$$|\varphi_{\nu+1}|_{k+\alpha}(t) << A(t).$$

Since $\varphi^{\nu+1}(t) = \varphi^\nu(t) + \varphi_{\nu+1}(t)$, by induction hypothesis, we have

$$|\varphi^{\nu+1}|_{k+\alpha}(t) << A(t).$$

□

As the radius of convergence of the power series $\sum_{\nu=1}^{\infty} \frac{s^\nu}{\nu^2}$ is equal to 1, $A(t)$ converges absolutely for $t \in \Delta_\varepsilon$ if $0 < \varepsilon < \frac{1}{mc}$. Therefore by Theorem2.5.20, $\varphi(t)$ converges with respect to the Hölder norm $| \ |_{k+\alpha}$ for $t \in \Delta_\varepsilon$. Consequently, $\varphi(t)$ is a C^k vector $(0,1)-$form on $M \times \Delta_\varepsilon$. Since $c = K_1 K_2 c_1 b$ depends on k, it does not follow immediately that there is an $\varepsilon > 0$ such that $\varepsilon \leq \frac{1}{mc}$ for all k. Hence we must prove C^∞ differentiability of $\varphi(t)$ in another way. We refer the proof to [2].

STEP2 : Proof of the Fact that$\{M_{\varphi(t)} \ : \ t \in \Delta_\varepsilon\}$ is a Complex Analytic Family.

We consider $\varphi = \varphi(t) = \sum_{\lambda,\mu} \varphi^\lambda_{\bar{\nu}}(z,t) d\bar{z}^\nu \partial_\lambda$ as a C^∞ vector $(0,1)-$form on $M \times \Delta_\varepsilon$:

$$\varphi = \varphi(t) = \sum_{\lambda=1}^{n} \left(\sum_{\nu=1}^{n} \varphi^\lambda_{\bar{\nu}} d\bar{z}^\nu + \sum_{\mu=1}^{m} \varphi^\lambda_{n+\mu} d\bar{t_\mu} \right) \frac{\partial}{\partial z^\lambda} + \sum_{\mu=1}^{m} \varphi^{n+\mu} \frac{\partial}{\partial t_\mu} \tag{2.54}$$

with $\varphi^\lambda_{n+\mu} = \varphi^{n+\mu} = 0$ for $\mu = 1, \ldots, m$.

Applying $\bar{\partial}$ on (2.54), we get

$$\bar{\partial}\varphi = \sum_{\lambda,\nu=1}^{n} \left(\sum_{\beta=1}^{n} \frac{\partial \varphi^\lambda_{\bar{\nu}}}{\partial \bar{z^\beta}} d\bar{z^\beta} + \sum_{\mu=1}^{m} \frac{\partial \varphi^\lambda_{\bar{\nu}}}{\partial \bar{t_\mu}} d\bar{t_\mu} \right) \wedge d\bar{z}^\nu \frac{\partial}{\partial z^\lambda}.$$

Since $\varphi^\lambda_{\bar{\nu}} = \varphi^\lambda_{\bar{\nu}}(z,t)$ are holomorphic in $t_1, \ldots, t_m$, we have

$$\bar{\partial}\varphi = \bar{\partial}\varphi(t),$$

where $\bar{\partial}\varphi(t)$ denotes the exterior differential of $\varphi(t)$ as a vector $(0,1)-$form on M with t fixed. Similarly, we obtain

$$[\varphi, \varphi] = [\varphi(t), \varphi(t)].$$

Therefore, as a C^∞ vector $(0,1)$–form on the complex manifold $M \times \Delta_\varepsilon$, φ satisfies the integrability condition

$$\bar{\partial}\varphi = \frac{1}{2}[\varphi, \varphi].$$

Since $\varphi(z,0) = 0$, by Theorem2.5.10, φ defines a complex structure $\mathcal{M}$ on the differentiable manifold $M \times \Delta_\varepsilon$ provided that Δ_ε is sufficiently small. Now, we want to show that there exists a holomorphic map $\hat{\omega}$ from $\mathcal{M}$ onto Δ_ε such that $\hat{\omega}^{-1}(t) = M_{\varphi(t)}$ and the rank of the Jocobian of $\hat{\omega}$ is equal to m at every point of $\mathcal{M}$. Let

$$L_\nu = \begin{cases} \frac{\partial}{\partial \bar{z^\nu}} - \sum\limits_{\lambda=1}^{n} \varphi_{\bar{\nu}}^{\lambda}(z,t)\frac{\partial}{\partial z^\lambda} & \text{for} \quad 1 \le \nu \le n. \\ \frac{\partial}{\partial \bar{t}_{\nu-n}} & \text{for} \quad n+1 \le \nu \le n+m. \end{cases}$$

Then the partial differential equation

$$(\bar{\partial} - \varphi)f = 0$$

is reduced to the system of partial differential equations :

$$L_\nu f = 0, \qquad \nu = 1, \dots, n+m. \tag{2.55}$$

Since $\varphi(z,0) = 0$, $L_1, \dots, L_{n+m}$, $\overline{L_1}, \dots, \overline{L_{n+m}}$ are linearly independent on $M \times \Delta_\varepsilon$, where Δ_ε is chosen to be sufficiently small. Let $\{U_j\}$ be a sufficiently fine locally finite open covering of M. Then, by Newlander-Nirenberg Theorem2.5.9 and Theorem2.5.3, there exists local complex coordinates :

$$\left(\zeta_j^1(z,t), \cdots, \zeta_j^{n+m}(z,t)\right)$$

of $\mathcal{M}$ on each $U_j \times \Delta_\varepsilon$. Clearly, we may assume that

$$\left(\zeta_j^{n+1}(z,t), \cdots, \zeta_j^{n+m}(z,t)\right) = (t_1, \cdots, t_m).$$

Therefore, the map $\hat{\omega} : \mathcal{M} \to \Delta_\varepsilon$ given by

$$\hat{\omega}\left(\zeta_j^1(z,t), \cdots, \zeta_j^n(z,t), t_1, \cdots, t_m\right) = (t_1, \cdots, t_m)$$

is holomorphic and the rank of the Jocobian of $\hat{\omega}$ is equal to m at every point of $\mathcal{M}$. Moreover, it is clear that $\hat{\omega}^{-1}(t) = M_{\varphi(t)}$. Thus, $\{M_{\varphi(t)} : t \in \Delta_\varepsilon\}$ forms a complex analytic family $(\mathcal{M}, \Delta_\varepsilon, \hat{\omega})$. Therefore, the proof of Theorem of Existence 2.5.1 is completed.

Finally, we state a deep theorem to end this chapter.

Theorem 2.5.21 *(**General Theorem of Existence**)[9] For any compact complex manifold M, there exists a complete complex analytic family $\{M_t, t \in B\}$ with $0 \in B \subset \mathbb{C}^m$ and $M_0 = M$.*

We will give the definition of a complete complex analytic family in next chapter.

Chapter 3

Comparison between the Number of Moduli $m(M)$ and $\dim H^1(M, \Theta)$

3.1 Number of Moduli of Compact Complex Manifold

Let $(\mathcal{M}, B, \hat{\omega})$ be a complex analytic family of compact complex manifolds, where B is a domain of $\mathbb{C}^m$ containing 0.

Definition 3.1.1 *$(\mathcal{M}, B, \hat{\omega})$ is said to be an effectively parametrized complex analytic family if the $\mathbb{C}$-linear map :*

$$\rho_t \ : \ T_t(B) \to H^1(M_t, \Theta_t)$$

is injective. In this case, we say that the parameter $t = (t_1, \ldots, t_m) \in B \subset \mathbb{C}^m$ *is effective.*

Definition 3.1.2 *$(\mathcal{M}, B, \hat{\omega})$ is said to be complete at $0 \in B$ if for any complex analytic family $(\mathcal{N}, D, \pi)$ such that D is a domain of $\mathbb{C}^l$ containing 0 and $\pi^{-1}(0) = \hat{\omega}^{-1}(0)$, there exists a domain E with $0 \in E \subset D$, and a holomorphic map $h : s \to t = h(s)$ with $h(0) = 0$ such that $(\mathcal{N}_E, E, \pi)$ is the complex analytic family induced from $(\mathcal{M}, B, \hat{\omega})$ by h. Loosely speaking, if $(\mathcal{M}, B, \hat{\omega})$ is complete at $0 \in B$, then $(\mathcal{M}, B, \hat{\omega})$ contains all sufficiently small deformations of $M_0 = \hat{\omega}^{-1}(0)$.*

Definition 3.1.3 *$(\mathcal{M}, B, \hat{\omega})$ is said to be a complete complex analytic family if $(\mathcal{M}, B, \hat{\omega})$ is complete at every point $t \in B$.*

Definition 3.1.4 *Let M be a compact complex manifold. If there exists an effectively parametrized and complete complex analytic family $(\mathcal{M}, B, \hat{\omega})$ with $\hat{\omega}^{-1}(0) = M$, then we define the number of moduli $m(M)$ of M by*

$$m(M) = dim B = m.$$

Otherwise, we do not define the number of moduli of M.

Claim: $m(M)$ is independent of the choice of complex analytic family used for the definition.

Proof : Let $(\mathcal{M}, B, \hat{\omega})=\{M_t = \hat{\omega}^{-1}(t) : t \in B\}$ and $(\mathcal{N}, D, \pi) = \{N_s = \pi^{-1}(s) : s \in D\}$ be effectively parametrized and complete complex analytic families with $\hat{\omega}^{-1}(0) = \pi^{-1}(0) = M$. As $(\mathcal{M}, B, \hat{\omega})$ is complete, there exists a domain E of D and a holomorphic map $h : s \to t = h(s)$ from E to B such that $(\mathcal{N}_E, E, \pi)$ is the complex analytic family induced from $(\mathcal{M}, B, \hat{\omega})$ by h. Then, by Theorem 1.5.2, we have

$$\frac{\partial N_s}{\partial s_\mu} = \frac{\partial M_{h(s)}}{\partial s_\mu} = \sum_{\lambda=1}^{m} \frac{\partial t_\lambda}{\partial s_\mu} \frac{\partial M_t}{\partial t_\lambda}, \qquad \mu = 1, \ldots, l. \tag{3.1}$$

where $(t_1, \ldots, t_m) = h(s)$. Since $(\mathcal{N}, D, \pi)$ is effectively parametrized,

$$\left\{ \frac{\partial N_s}{\partial s_1}, \ldots, \frac{\partial N_s}{\partial s_l} \right\}$$

is linearly independent. Then by (3.1), we obtain $m \geq l$. Similarly, we have $l \geq m$. □

Observe that the effectively parametrized and complete complex analytic family $(\mathcal{M}, B, \hat{\omega})$ with $\hat{\omega}^{-1}(0) = M$ is uniquely determined by M provided that the domain B of $\mathbb{C}^m$ containing 0 is sufficiently small.

Theorem 3.1.5 *(***Theorem of Completeness***)[6]*
If $\rho_0 : T_0(B) \to H^1(M_0, \Theta_0)$ is surjective, then $(\mathcal{M}, B, \hat{\omega})$ is complete at $0 \in B$.

The proof of the above theorem will be given in the next chapter.

Theorem 3.1.6 *Suppose that $H^2(M, \Theta) = 0$ and $m(M)$ is defined. Then $m(M) = dim H^1(M, \Theta)$.*

Proof : As $H^2(M, \Theta) = 0$, by Theorem2.5.1(Theorem of Existence), there exists a complex analytic family $(\mathcal{M}, B, \hat{\omega})$=$\{M_t = \hat{\omega}^{-1}(t) : t \in B\}$ with $0 \in B \subset \mathbb{C}^m$ satisfying the following conditions :

$$(*) \begin{cases} \text{(i)} \quad \hat{\omega}^{-1}(0) = M, \\ \text{(ii)} \ \rho_0 : \frac{\partial}{\partial t} \to \left(\frac{\partial M_t}{\partial t_\lambda}\right)_{t=0} \ \text{is an isomorphism of } T_0(B) \text{ onto } H^1(M, \Theta). \end{cases}$$

By the condition (ii), we see from the proof of the Claim that $dimH^1(M, \Theta) \leq m(M)$. On the other hand, it is clear that $m(M) \leq dimH^1(M, \Theta)$. $\square$

In the following, the complex analytic family $(\mathcal{M}, B, \hat{\omega})$ is assumed to be satisfying the condition $(*)$.

Theorem 3.1.7 *Assume that $H^2(M, \Theta) = 0$. Then $m(M)$ is defined if and only if $dimH^1(M_t, \Theta_t)$ is independent of $t \in B$ provided that the domain B of $\mathbb{C}^m$ containing 0 is sufficiently small. Moreover, if this condition is satisfied, then $(\mathcal{M}, B, \hat{\omega})$ is an effectively parametrized and complete complex analytic family.*

Proof : Suppose that $m(M)$ is defined, then there exists an effectively parametrized and complete complex analytic family $(\mathcal{N}, D, \pi)$ with $\pi^{-1}(0) = M$ such that $(\mathcal{M}, B, \hat{\omega})$ is the complex analytic family induced from a holomorphic map $h : B \to D$ with $h(0) = 0$. Moreover, by Theorem3.1.6, we have

$$m = dimB = dimH^1(M, \Theta) = m(M) = dimD = l.$$

Since D is an effective parameter space, we have

$$dimH^1(N_s, \Theta_s) \geq l = m.$$

As $(\mathcal{M}, B, \hat{\omega})$ is induced from $(\mathcal{N}, D, \pi)$, it follows that

$$dimH^1(M_t, \Theta_t) \geq m.$$

Therefore, by Theorem1.3.6(the upper-semicontinuity theorem), we conclude that $dimH^1(M_t, \Theta_t) = m$.

Conversely, suppose that $dimH^1(M_t, \Theta_t) = dimH^1(M, \Theta) = m$ for $t \in B$. As $\rho_0 : T_0(B) \to H^1(M, \Theta)$ is an isomorphism,

$$\left\{ \left(\frac{\partial M_t}{\partial t_1}\right)_{t=0}, \dots, \left(\frac{\partial M_t}{\partial t_m}\right)_{t=0} \right\}$$

are linearly independent. Then it follows that

$$\left\{ \frac{\partial M_t}{\partial t_1}, \dots, \frac{\partial M_t}{\partial t_m} \right\}$$

are linearly independent for $t \in B$ as B is chosen to be sufficiently small.[5] Therefore, the linear map

$$\rho_t : T_t(B) \to H^1(M_t, \Theta_t)$$

is an isomorphism for all $t \in B$. By Theorem3.1.5(Theorem of Completeness), $(\mathcal{M}, B, \hat{\omega})$ is an effectively parametrized and complete complex analytic family. Thus, $m(M)$ is defined. $\square$

Now, we state a useful theorem and refer the proof to [5].

Theorem 3.1.8 *If $H^0(M, \Theta) = H^2(M, \Theta) = 0$, then $m(M)$ is defined and is equal to $dimH^1(M, \Theta)$.*

3.2 Examples

(i)**Projective Space $\mathbb{P}^n$**

By results of Bott[1], it is well known that

$$m(\mathbb{P}^n) = \dim H^1(\mathbb{P}^n, \Theta) = 0.$$

(ii) **Complex Tori**

Let $M = \mathbb{C}^n/G$ be a complex torus, where $G = \{\sum_{j=1}^{2n} m_j w_j : m_j \in \mathbb{Z}\}$ and $\{w_1, \ldots, w_{2n}\} \subset \mathbb{C}^n$ are linearly independent over $\mathbb{R}$.

Claim : $\dim H^1(M, \Theta) = \dim \mathbb{H}^{0,1}(T(M)) = n^2$.

Proof : Let $(z^1, \ldots, z^n)$ be the coordinate of $\mathbb{C}^n$, then it is clear that $\{\frac{\partial}{\partial z^1}, \ldots, \frac{\partial}{\partial z^n}\}$ is a basis for $T(M)$ and $\{d\bar{z}^1, \ldots, d\bar{z}^n\}$ is a basis for $\bar{T}^*(M)$. Hence for any $\varphi \in \mathcal{L}^{0,1}(T(M)) = \Gamma(M, \mathcal{A}(\bar{T}^*(M) \otimes T(M))$, it can be represented as

$$\varphi = \sum_{\alpha,\nu=1}^{n} \varphi^\alpha_\nu d\bar{z}^\nu \otimes \frac{\partial}{\partial z^\alpha}.$$

We define a Hermitian metric on M by $\sum_{\alpha,\beta=1}^{n} g_{\alpha\bar{\beta}} dz^\alpha \otimes d\bar{z}^\beta$, where $g_{\alpha\bar{\beta}} = \delta_{\alpha\beta}$. Then the Laplace Beltrami operator $\Box$ for this metric is given by [10]:

$$\Box = -\sum_{\alpha=1}^{n} \frac{\partial^2}{\partial z^\alpha \partial \bar{z}^\alpha}.$$

Further, if we define a Hermitian metric on $T(M)$ by $\sum_{\alpha=1}^{n} \zeta^\alpha \bar{\zeta}^\alpha$, $(\zeta^1, \ldots, \zeta^n)$ are the fibre coordinates of $T(M)$, then the ajoint operator ϑ_I of $\bar{\partial}$ and the complex Laplacian $\Box_I$ with respect to this metric is given by[10]:

$$\vartheta_I = \vartheta \quad \text{and} \quad \Box_I = \Box = -\sum_{\alpha=1}^{n} \frac{\partial^2}{\partial z^\alpha \partial \bar{z}^\alpha}$$

respectively. Therefore, $\Box_I \varphi = 0$ means that

$$\Box \varphi^\alpha_{\bar{\nu}} = -\sum_{\alpha=1}^{n} \frac{\partial^2}{\partial z^\alpha \partial \bar{z}^\alpha} \varphi^\alpha_{\bar{\nu}} = 0$$

for all $\varphi^\alpha_{\bar{\nu}}$. As

$$(\Box \varphi^\alpha_{\bar{\nu}}, \varphi^\alpha_{\bar{\nu}}) = \left((\bar{\partial}\vartheta + \vartheta\bar{\partial})\varphi^\alpha_{\bar{\nu}}, \varphi^\alpha_{\bar{\nu}}\right) = \|\vartheta \varphi^\alpha_{\bar{\nu}}\|^2 + \|\bar{\partial} \varphi^\alpha_{\bar{\nu}}\|^2,$$

we have

$$\bar{\partial}\varphi^{\alpha}_{\bar{\nu}} = 0.$$

Since M is compact, $\varphi^{\alpha}_{\bar{\nu}} = c^{\alpha}_{\bar{\nu}}$ is a constant. Therefore, we get

$$\begin{aligned} \mathbb{H}^{0,1}(T(M)) &= \{\varphi \in \mathcal{L}^{0,1}(T(M)) : \Box\varphi = 0\} \\ &= \left\{ \sum_{\alpha,\nu=1}^{n} c^{\alpha}_{\bar{\nu}} d\bar{z^{\nu}} \otimes \tfrac{\partial}{\partial z^{\alpha}} : c^{\alpha}_{\bar{\nu}} \in \mathbb{C} \right\}. \end{aligned}$$

This implies that $\dim\mathbb{H}^{0,1}(T(M)) = n^2$. Then by (2.40), we get $\dim H^1(M,\Theta)$ $=\dim\mathbb{H}^{0,1}(T(M)) = n^2$. □

Using the suitable coordinates of $\mathbb{C}^n$, we may assume that the period matrix of M is of the form :

$$\begin{pmatrix} t^1_1 & \cdots & t^1_n \\ \cdots & \cdots & \cdots \\ t^1_n & \cdots & t^n_n \\ 1 & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & 1 \end{pmatrix}$$

Put $t = \left(t^{\beta}_{\alpha}\right)_{\alpha,\beta=1,\ldots,n}$ and write M_t for M. Let $w_j = \left(w^1_j(t), \ldots, w^n_j(t)\right)$ be the jth row of the period matrix. Then we have

$$w^{\beta}_j(t) = \begin{cases} t^{\beta}_j, & j = 1, \ldots, n. \\ \delta^{\beta}_{j-n}, & j = n+1, \ldots, 2n. \end{cases}$$

and $M_t = \mathbb{C}^n/G_t$, where $G_t = \{\sum_{j=1}^{2n} m_j w_j(t) : m_j \in \mathbb{Z}\}$. Note that

$$\{w_1(t), \ldots, w_{2n}(t)\}$$

are linearly independent over $\mathbb{R}$ if and only if $\det\left(\mathrm{Im}t_\alpha^\beta\right)_{\alpha,\beta=1,\dots,n} \neq 0$. Let

$$B = \{t : \det(\mathrm{Im}t_\alpha^\beta) > 0\}.$$

Then $\{M_t : t \in B\}$ forms a complex analytic family . In fact, let

$$\mathcal{G} = \left\{\left(\sum_{j=1}^{2n} m_j w_j(t), t\right) \in \mathbb{C}^n \times B : m_j \in \mathbb{Z}, t \in B\right\}.$$

Then $\mathcal{M} = \mathbb{C}^n \times B/\mathcal{G}$ is a complex manifold and the projection $(z,t) \to t$ of $\mathbb{C}^n \times B$ onto B induces a holomorphic map $\hat{\omega}$ of $\mathcal{M}$ onto B with $\hat{\omega}^{-1}(t) = \mathbb{C}^n \times t/G_t = M_t$. Hence $(\mathcal{M}, B, \hat{\omega})$=$\{M_t : t \in B\}$ is a complex analytic family.

Let Π be the canonical holomorphic map of $\mathbb{C}^n \times B$ onto $\mathcal{M}$. We choose a locally finite open covering $\{\mathcal{U}_k\}$ of $\mathcal{M}$ such that $\Pi^{-1}(\mathcal{U}_k)$ consists of infinitely many mutually disjoint domains $\mathcal{U}_{k_1}, \mathcal{U}_{k_2}, \dots$ of $\mathbb{C}^n \times B$. Since Π maps each $\mathcal{U}_{k_m}$ biholomorphically onto $\mathcal{U}_k$, we may use the coordinates of one of them as the coordinates of $\mathcal{U}_k$. Let $(z_k, t) = (z_k^1, \dots, z_k^n, t)$ be the coordinates on $\mathcal{U}_k$ obtained in this way. Then on each $\mathcal{U}_j \cap \mathcal{U}_k \neq \emptyset$, we have

$$z_i = z_k + \sum_{j=1}^{2n} m_{ik}^j w_j(t), \qquad m_{ik}^j \in \mathbb{Z}. \tag{3.2}$$

That is

$$f_{ik}^\beta(z_k, t) = z_i^\beta = z_k^\beta + m_{ik}^{n+\beta} + \sum_{\alpha=1}^n m_{ik}^\alpha t_\alpha^\beta. \tag{3.3}$$

Let

$$\theta_{ik}(t) = \sum_{\gamma=1}^n \frac{\partial f_{ik}^\gamma(z_k, t)}{\partial t_\alpha^\beta} \frac{\partial}{\partial z_i^\gamma} = m_{ik}^\alpha \frac{\partial}{\partial z_i^\beta}. \tag{3.4}$$

Then $\frac{\partial M_t}{\partial t^\beta_\alpha} = \theta(t) \in H^1(M_t, \Theta_t)$ is the cohomology class of the 1−cocycle $\{\theta_{ik}(t)\}$. Let $\Psi_t = \mathcal{A}(T(M_t))$, then we have the following exact sequence of sheaves :

$$0 \to \Theta_t \xrightarrow{\imath} \Psi_t \xrightarrow{\bar{\partial}} \bar{\partial}\Psi_t \to 0.$$

Since Ψ_t is a fine sheaf, we have the exact sequence of cohomology groups :

$$0 \to H^0(M_t, \Theta_t) \xrightarrow{\imath} H^0(M_t, \Psi_t) \xrightarrow{\bar{\partial}} H^0(M_t, \bar{\partial}\Psi_t) \xrightarrow{\delta_*} H^1(M_t, \Theta_t) \to 0.$$

Hence, we get

$$H^1(M_t, \Theta_t) \cong \frac{H^0(M_t, \bar{\partial}\Psi_t)}{\bar{\partial}H^0(M_t, \Psi_t)}.$$

Then by (2.39), we have

$$H^1(M_t, \Theta_t) \cong \frac{H^0(M_t, \bar{\partial}\Psi_t)}{\bar{\partial}H^0(M_t, \Psi_t)} \cong \mathbb{H}^{0,1}(T(M_t)). \tag{3.5}$$

Now, we want to show that

$$\left\{\rho_t\left(\frac{\partial}{\partial t^\beta_\alpha}\right) \ : \ \alpha, \beta = 1, \dots, n\right\}$$

are linearly independent, and then to conclude that $m(M_t)$=dim$H^1(M_t, \Theta_t)$=n^2. By (3.3), we get

$$\sum_{\alpha=1}^{n} m^\alpha_{ik}(t^\beta_\alpha - \overline{t^\beta_\alpha}) = (\overline{z^\beta_k} - z^\beta_k) - (\overline{z^\beta_i} - z^\beta_i). \tag{3.6}$$

Let $\left(u^\beta_\alpha\right)_{\alpha,\beta=1,\dots,n}$ be the inverse matrix of $\left(t^\beta_\alpha - \overline{t^\beta_\alpha}\right)_{\alpha,\beta=1,\dots,n}$, then we have

$$m^\alpha_{ik} = \sum_{\gamma=1}^{n} \left(\overline{z^\gamma_k} - z^\gamma_k\right) u^\alpha_\gamma - \sum_{\gamma=1}^{n} \left(\overline{z^\gamma_i} - z^\gamma_i\right) u^\alpha_\gamma. \tag{3.7}$$

Putting

$$\Psi_k = \sum_{\gamma=1}^{n} \left(\overline{z^\gamma_k} - z^\gamma_k\right) u^\alpha_\gamma \frac{\partial}{\partial z^\beta},$$

we get

$$\theta_{ik}(t) = \Psi_k - \Psi_i. \tag{3.8}$$

Since $\theta_{ik}(z_k, t)$ is holomorphic in z_k, we obtain

$$\bar{\partial}\Psi_k(t) = \bar{\partial}\Psi_i(t).$$

Hence by the definition of δ^*, we have

$$\delta^*\varphi(t) = \theta(t), \qquad \text{where } \varphi(t) = \bar{\partial}\Psi_k(t) \text{ on } \mathcal{U}_k \cap M_t.$$

As

$$\bar{\partial}\overline{z_k^\gamma} = (\bar{\partial} + \partial)\overline{z^\gamma} = d\overline{z^\gamma} \quad \text{and} \quad \bar{\partial}z_k^\gamma = 0,$$

we obtain

$$\varphi(t) = \bar{\partial}\Psi_k = \sum_{\gamma=1}^n u_\gamma^\alpha d\overline{z^\gamma} \otimes \frac{\partial}{\partial z^\beta} \in \mathbb{H}^{0,1}(T(M_t)). \tag{3.9}$$

Thus via the isomorphism (3.5),

$$\rho_t\left(\frac{\partial}{\partial t_\alpha^\beta}\right) = \theta(t) \quad \text{corresponds to} \quad \sum_{\gamma=1}^n u_\gamma^\alpha d\overline{z^\gamma} \otimes \frac{\partial}{\partial z^\beta}.$$

Since $\left(u_\gamma^\alpha\right)_{\alpha,\gamma=1,\ldots,n}$ is nonsingular, $\rho_t\left(\frac{\partial}{\partial t_\alpha^\beta}\right)$, $\alpha, \beta = 1, \ldots, n$, are linearly independent. As $\dim H^1(M_t, \Theta_t){=}n^2$, $\rho_t : T_t(B) \to H^1(M_t, \Theta_t)$ is an isomorphism, by Theorem of Completeness 3.1.5, $m(M_t)$ is defined and $m(M_t){=}\dim H^1(M_t, \Theta_t) = n^2$.

(iii)**An Example of Hirzebruch**

Let $\zeta = \frac{\zeta_1}{\zeta_0}$ be the inhomogeneous coordinates on $\mathbb{P}^1$. Putting $\zeta = \infty$ for

$\zeta_0 = 0$, we consider $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$. Let

$$\begin{aligned} M &= \mathbb{P}^1 \times \mathbb{P}^1 = \{(z,\zeta) : z,\zeta \in \mathbb{P}^1\}, \\ S &= 0 \times \mathbb{P}^1 \subset M, \\ W &= U_\varepsilon \times \mathbb{P}^1, \\ \hat{S} &= 0 \times \mathbb{P}^1, \\ \hat{W} &= U_\varepsilon \times \mathbb{P}^1, \end{aligned}$$

where $U_\varepsilon = \{z : |z| < \varepsilon\}$. Fix an integer $m > 0$ and define $\Phi : \hat{W} - \hat{S} \to W - S$ as follows :

$$\Phi : (z, \hat{\zeta}) \to (z, \zeta) = (z, \frac{\hat{\zeta}}{z^m}).$$

Clearly, Φ is biholomorphic on $\hat{W} - \hat{S}$. Let

$$\hat{M}_m = (M - S) \cup \hat{W},$$

where $(z,\zeta) = (z,\hat{\zeta})$ if $\hat{\zeta} = z^m\zeta,\ 0 < |z| < \varepsilon$. Then we have the following facts[Hirzebruch] :

(i). $\hat{M}_m$ and $\hat{M}_n$ are not biholomorphic if $m \neq n$.

(ii). $\hat{M}_{2m}$ is homeomorphic to M.

(iii). $\hat{M}_{2m+1}$ is homeomorphic to $\hat{M}_1$.

(iv). $\hat{M}_1$ is not homeomorphic to M.

Proof of (iv) : It suffices to prove that the homology intersection properties of M and $\hat{M}_1$ are distinct. Since $\mathbb{P}^1$ is homeomorphic to S^2, $M = S^2 \times S^2$. Therefore, a basis for $H_2(M,\mathbb{Z})$ is given by $\{S,T\}$, where $S = \{0\} \times S^2$, $T = S^2 \times \{0\}$. Hence, any 2−cycle C is homologous ($\sim$) to $aS + bT$, $a, b \in \mathbb{Z}$. We denote the intersection multiplicity of two cycles S and T by $I(S,T)$.

Since $S \sim S_1 = \{1\} \times S^2$, and S does not intersect with S_1, we have

$$I(S,S) = I(S,S_1) = 0.$$

Similarly we have $I(T,T) = 0$. Since S and T intersect transversally at the unique point $\{0\} \times \{0\}$,

$$I(S,T) = I(T,S) = 1.$$

Hence, we get

$$\begin{aligned} I(C,C) &= I(aS+bT, aS+bT) \\ &= a^2 I(S,S) + b^2 I(T,T) + 2ab I(S,T) \\ &\equiv 0 \pmod 2. \end{aligned}$$

On $\hat{M}_1 = (M - S) \cup \hat{W}$, $(z,\zeta) \in M - S$ with $0 < |z| < \varepsilon$ is identified with $(z,\hat{\zeta}) = (z, z\zeta) \in \hat{W}$. Consequently, for any $t \in \mathbb{C}$,

$$C_t = \{(z,t) \in M - S\} \bigcup \{(z,zt) \in \hat{W}\}$$

is a complex submanifold of $\hat{M}_1$. Since C_t depends continuously on t, $C_t \sim C_0$. On the other hand, C_t and C_0 intersect transversally at the unique point $(0,0) \in \hat{W}$. Therefore,

$$I(C_0,C_0) = I(C_t,C_0) = 1.$$

Since for any 2−cycle C on M, $I(C,C) \equiv 0 \pmod 2$, we conclude that $\hat{M}_1$ and M are topologically different. □

Proof of (ii) and (iii) : Let $U_1 = \mathbb{C}$ and $U_2 = \mathbb{P}^1 - \{0\}$, then $\mathbb{P}^1 = \mathbb{C} \cup$

$\{\infty\}=U_1 \cup U_2$. Let z_1 and z_2 be the non-homogeneous coordinates on U_1 and U_2 respectively. Then on $U_1 \cup U_2$, we have

$$z_1 z_2 = 1.$$

Since $M - S = U_2 \times \mathbb{P}^1$, taking $\varepsilon = \infty$, we have

$$\hat{M}_m = \left(U_2 \times \mathbb{P}^1\right) \bigcup \left(U_1 \times \mathbb{P}^1\right),$$

where $(z_1, \zeta_1) \in U_1 \times \mathbb{P}^1$ and $(z_2, \zeta_2) \in U_2 \times \mathbb{P}^1$ are the same point on $\hat{M}_m$ if

$$z_1 z_2 = 1 \quad \text{and} \quad \zeta_1 = z^m \zeta_2. \tag{3.10}$$

Note that $\zeta_1 = \frac{1}{\zeta}$, $\zeta_2 = \frac{1}{\zeta}$ and $\frac{1}{z_2} = z$.

Now, for any fixed positive integer $k \leq \frac{m}{2}$, we define M_t as

$$M_t = \left(U_1 \times \mathbb{P}^1\right) \bigcup \left(U_2 \times \mathbb{P}^1\right),$$

where $(z_1, \zeta_1) \in U_1 \times \mathbb{P}^1$ and $(z_2, \zeta_2) \in U_2 \times \mathbb{P}^1$ are the same point of M_t if

$$z_1 z_2 = 1 \quad \text{and} \quad \zeta_1 = z_2^m \zeta_2 + t z_2^k. \tag{3.11}$$

Then, it is clear that $\{M_t : t \in \mathbb{C}\}$ forms a complex analytic family with $M_0 = \hat{M}_m$.

<u>**Claim**</u> : For $t \neq 0$, $M_t = \hat{M}_{m-2k}$.

Proof : Let

$$\begin{aligned} (z_1, \zeta_1') &= \left(z_1, \frac{z_1^k \zeta_1 - t}{t\zeta_1}\right), \\ (z_2, \zeta_2') &= \left(z_2, \frac{\zeta}{t z_2^{m-k} \zeta_2 + t^2}\right). \end{aligned}$$

Then since

$$\det\begin{pmatrix} z_1^k & -t \\ t & 0 \end{pmatrix} = t^2 \neq 0 \quad \text{and} \quad \det\begin{pmatrix} 1 & 0 \\ tz_2^{m-k} & t^2 \end{pmatrix} = t^2 \neq 0,$$

(z_i, ζ_i') define a new coordinates on $U_i \times \mathbb{P}^1,\ i = 1, 2$. By (3.11), we have

$$\begin{aligned} \zeta_1' &= \tfrac{z_1^k \zeta_1 - t}{t\zeta_1} \\ &= \tfrac{z_1^k(z_2^m \zeta_2 + tz_2^k) - t}{t(z_2^m \zeta_2 + tz_2^k)} \\ &= \tfrac{z_2^{m-k}\zeta_2}{tz_2^m\zeta_2 + t^2 z_2^k} \\ &= z_2^{m-2k} \tfrac{\zeta_2}{tz_2^{m-k}\zeta_2 + t^2} \\ &= z_2^{m-2k}\zeta_2'. \end{aligned}$$

Therefore, we have

$$M_t = \hat{M}_{m-2k}, \tag{3.12}$$

that is, M_t does not change its complex structure for all $t \neq 0$. □

From this Claim, we obtain the following results :

(a). $\hat{M}_{2m}$ is a deformation of $\hat{M}_0 = \mathbb{P}^1 \times \mathbb{P}^1 = M$.

(b). $\hat{M}_{2m+1}$ is a deformation of $\hat{M}_1$.

Hence by Theorem 1.1.3,

$\hat{M}_{2m}$ is diffeomorphic to($\cong$) M and $\hat{M}_{2m+1} \cong \hat{M}_1$. □

Proof of (i) : We prove (i) by computing the number of linearly independent holomorphic vector fields on $\hat{M}_m,\ m \geq 0$.

Lemma 3.2.1 *Any holomorphic vector field on* $\mathbb{C} \times \mathbb{P}^1$ *is of the form :*

$$v = f(z)\frac{\partial}{\partial z} + \left(a(z)\zeta^2 + b(z)\zeta + c(z)\right)\frac{\partial}{\partial \zeta}, \tag{3.13}$$

where ζ *is a non-homogeneous coordinate on* $\mathbb{P}^1$ *and* f, a, b, c *are holomorphic functions in* $z \in \mathbb{C}$.

Proof : Let $\eta = \frac{1}{\zeta}$ be the local coordinate on $\mathbb{P}^1$ at $\zeta = \infty$. Restrict the vector field to $\mathbb{C} \times (\mathbb{P}^1 - \{\infty\}) = \mathbb{C}^2$:

$$v = f(z,\zeta)\frac{\partial}{\partial z} + g(z,\zeta)\frac{\partial}{\partial \zeta},$$

where f and g are holomorphic on $\mathbb{C}^2$. At ∞, we have

$$v = h(z,\eta)\frac{\partial}{\partial z} + k(z,\eta)\frac{\partial}{\partial \eta},$$

where $h,\ k$ are holomorphic. Since $\zeta = \frac{1}{\eta}$, we have

$$\frac{\partial}{\partial \eta} = -\zeta^2\frac{\partial}{\partial \zeta}.$$

Hence at ∞, we get

$$v = h(z,\eta)\frac{\partial}{\partial z} - \zeta^2 k(z,\eta)\frac{\partial}{\partial \zeta}.$$

Since $f(z,\zeta) = h(z,\eta)$, $f(z,\zeta)$ is holomorphic on $\mathbb{C} \times \mathbb{P}^1$. Thus $f(z,\zeta)$ is constant as a function of ζ :

$$f(z,\zeta) = f(z).$$

Finally, $g(z,\zeta) = -\zeta^2 k(z,\eta)$ implies that $g(z,\zeta)$ has a pole of order ≤ 2 at ∞. Therefore we have

$$g(z,\zeta) = a(z)\zeta^2 + b(z)\zeta + c(z).$$

□

Consequently, a holomorphic vector field on $\hat{M}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ is given by

$$(a_1 z_1^2 + b_1 z_1 + c_1)\frac{\partial}{\partial z_1} + (\alpha_1\zeta_1^2 + \beta_1\zeta_1 + \gamma_1)\frac{\partial}{\partial \zeta_1},$$

where $a_1,\ b_1\ c_1,\ \alpha_1,\ \beta_1,\ \gamma_1 \in \mathbb{C}$ are constants. This implies that there are 6 linearly independent holomorphic vector fields on $\hat{M}_0$.

Next we consider holomorphic vector fields on $\hat{M}_m = (U_1 \times \mathbb{P}^1) \cup (U_2 \times \mathbb{P}^1),\ m \geq 1$. From Lemma 3.2.1, a holomorphic vector field on $U_i \times \mathbb{P}^1$ has the form :

$$v_i(z_i)\frac{\partial}{\partial z_i} + \left(\alpha_i(z_i)\zeta_i^2 + \beta_i(z_i)\zeta_i + \gamma_i(z_i)\right)\frac{\partial}{\partial \zeta_i},\ i = 1, 2, \tag{3.14}$$

where $v_i(z_i),\ \alpha_i(z_i),\ \beta_i(z_i),\ \gamma_i(z_i)$ are entire functions of z_i. Clearly, we have

$$\begin{array}{rl} & v_1(z_1)\frac{\partial}{\partial z_1} + (\alpha_1(z_1)\zeta_1^2 + \beta_1(z_1)\zeta_1 + \gamma_1(z_1))\frac{\partial}{\partial \zeta_1} \\ = & v_2(z_2)\frac{\partial}{\partial z_2} + (\alpha_2(z_2)\zeta_2^2 + \beta_2(z_2)\zeta_2 + \gamma_2(z_2))\frac{\partial}{\partial \zeta_2}. \end{array} \tag{3.15}$$

on $(U_1 \times \mathbb{P}^1) \cap (U_2 \times \mathbb{P}^1)$. Since $z_1 = \frac{1}{z_2}$ and $\zeta_1 = z_2^m \zeta_2$ by (3.10), we have

$$\left\{ \begin{array}{rcl} \frac{\partial}{\partial z_2} & = & -z_1^2\frac{\partial}{\partial z_1} + m z_1 \zeta_1 \frac{\partial}{\partial \zeta_1}, \\ \frac{\partial}{\partial \zeta_2} & = & \frac{1}{z_1^m}\frac{\partial}{\partial \zeta_1}. \end{array} \right.$$

Substituting these into (3.15), and comparing the coefficients, we get

$$\left\{ \begin{array}{rcl} v_1(z_1) & = & -z_1^2 v_2\left(\frac{1}{z_1}\right), \\ \alpha_1(z_1) & = & z_1^m \alpha_2\left(\frac{1}{z_1}\right), \\ \beta_1(z_1) & = & m z_1 v_2\left(\frac{1}{z_1}\right) + \beta_2\left(\frac{1}{z_1}\right), \\ \gamma_1(z_1) & = & \frac{1}{z_1^m}\gamma_2\left(\frac{1}{z_1}\right). \end{array} \right.$$

From the above equalities, we have

$$\left\{ \begin{array}{rcl} v_1(z_1) & = & a z_1^2 + b z_1 + c, \\ \alpha_1(z_1) & = & \sum\limits_{k=0}^{m} c_k z_1^k, \\ \beta_1(z_1) & = & -m a z_1 + d, \\ \gamma_1(z_1) & = & 0. \end{array} \right.$$

This implies that there are $(m+5)$ linearly independent vector fields on $\hat{M}_m$, $m \geq 1$. Since $\hat{M}_1 \not\cong M = \hat{M}_0$ by (iv), we conclude that if $m \neq n$, then $\hat{M}_m$ and $\hat{M}_n$ are not biholomorphic. □

Note that the $\dim H^0(\hat{M}_m, \Theta)$ is the number of linearly independent holomorphic vector fields on $\hat{M}_m$, and is given by

$$\dim H^0(\hat{M}_m, \Theta) = \begin{cases} 6 & \text{for } m = 0, \\ m+5 & \text{for } m \geq 1. \end{cases} \tag{3.16}$$

Therefore, Theorem 3.1.8 can not apply to $\hat{M}_m$.

Next, we compute $\dim H^1(\hat{M}_m, \Theta)$.

Put $W_1 = U_1 \times \mathbb{P}^1$ and $W_2 = U_2 \times \mathbb{P}^1$. Then

$$\mathcal{W} = \{W_1,\ W_2\}$$

is an open covering of $\hat{M}_m$, and both W_1 and W_2 are biholomorphic to $\mathbb{C} \times \mathbb{P}^1$. Since $H^1(\mathbb{C} \times \mathbb{P}^1, \Theta) = 0$,

$$H^1(W_1, \Theta) = H^1(W_2, \Theta) = 0.$$

Therefore by Theorem 2.2.1, we have

$$H^1(\hat{M}_m, \Theta) = H^1(\mathcal{W}, \Theta) = Z^1(\mathcal{W}, \Theta)/\delta C^0(\mathcal{W}, \Theta).$$

We represent a $1-$cocycle $\{\theta_{12}, \theta_{21}\} \in Z^1(\mathcal{W}, \Theta)$ by the holomorphic vector field $\theta_{12} = -\theta_{21}$ on $W_1 \cap W_2$. Then, this $1-$cocycle belongs to $\delta C^0(\mathcal{W}, \Theta)$ if and only if there exists holomorphic vector fields θ_1 and θ_2 respectively on W_1 and W_2 such that

$$\theta_2 - \theta_1 = \theta_{12}. \tag{3.17}$$

By Lemma 3.2.1, $\theta_i,\ i=1,2$, are written in the form :

$$\theta_i = v_i(z_i)\frac{\partial}{\partial z_i} + \left(\alpha_i(z_i)\zeta_i^2 + \beta_i(z_i)\zeta_i + \gamma_i(z_i)\right)\frac{\partial}{\partial \zeta_i}.$$

Similarly, by Lemma 3.2.1, we write θ_{12} as follows in terms of the coordinates (z_1, ζ_1) :

$$\theta_{12} = v_{12}(z_1)\frac{\partial}{\partial z_1} + \left(\alpha_{12}(z_1)\zeta_1^2 + \beta_{12}(z_1)\zeta_1 + \gamma_{12}(z_1)\right)\frac{\partial}{\partial \zeta_1}.$$

In terms of the coordinates (z_1, ζ_1), (3.17) is reduced to the following system of equations :

$$\begin{cases} -z_1^2 v_2\left(\frac{1}{z_1}\right) - v_1(z_1) & = & v_{12}(z_1), \\ z_1^m \alpha_2\left(\frac{1}{z_1}\right) - \alpha_1(z_1) & = & \alpha_{12}(z_1), \\ m z_1 v_2\left(\frac{1}{z_1}\right) + \beta_2\left(\frac{1}{z_1}\right) - \beta_1(z_1) & = & \beta_{12}(z_1), \\ \frac{1}{z_1^m}\gamma_2\left(\frac{1}{z_1}\right) - \gamma_1(z_1) & = & \gamma_{12}(z_1). \end{cases}$$

For $m=0,\ 1$, these equations always have a solution.

For $m \geq 2$, let $\gamma_{12}(z_1) = \sum_{n=-\infty}^{\infty} c_n z_1^n$ be the Laurent expansion of $\gamma_{12}(z_1)$. Then the above equations have a solution if and only if

$$c_{-1} = c_{-2} = \cdots c_{-m+1} = 0.$$

Hence we have

$$\dim H^1(\hat{M}_m, \Theta) = \begin{cases} 0, & m = 0,\ 1. \\ m-1, & m = 2,\ 3, \cdots. \end{cases}$$

Finally, we compute $\dim H^2(\hat{M}_m, \Theta)$.

Since $\dim H^2(\hat{M}_m, \Theta) = \dim H^0(\hat{M}_m, \Omega^1(K))$, where $\Omega^1(K)$ is the sheaf of germs of holomorphic 1–forms over $\hat{M}_m$ with coefficients in the canonical bundle

$K = T^*(\hat{M}_m) \wedge T^*(\hat{M}_m)$, it suffices to compute $\dim H^0(\hat{M}_m, \Omega^1(K))$. Note that for any $\psi \in H^0(\hat{M}_m, \Omega^1(K))$, it can be represented in the form :

$$\psi = (g(z_1, \zeta_1)dz_1 + h(z_1, \zeta_1)d\zeta_1) \otimes (dz_1 \wedge d\zeta_1)$$

on $U_1 \times \mathbb{C} \subset U_1 \times \mathbb{P}^1$, where $g(z_1, \zeta_1)$ and $h(z_1, \zeta_1)$ are holomorphic functions of z_1 and $\zeta_1 \neq \infty$. We observe that ψ is required to be holomorphic in a neighbourhood of $U_1 \times \{\infty\}$. Changing the coordinate ζ_1 to the local coordinate $w = \frac{1}{\zeta_1}$ at ∞, we get

$$\psi = -\left(g(z_1, \frac{1}{w})dz_1 - h(z_1, \frac{1}{w})\frac{dw}{w^2}\right) \otimes \left(dz_1 \wedge \frac{dw}{w^2}\right).$$

Therefore, $\frac{1}{w^2}g(z_1, \frac{1}{w})$ and $\frac{1}{w^4}h(z_1, \frac{1}{w})$ must be holomorphic in w. This implies that

$$g(z_1, \zeta_1) = h(z_1, \zeta_1) = 0,$$

namely, $\psi = 0$. Therefore, we conclude that $\dim H^2(\hat{M}_m, \Theta) = 0$.

We summarize the previous results as follows :

$$\dim H^0(\hat{M}_m, \Theta) = \begin{cases} 6, & m = 0. \\ m+5, & m = 1,\ 2, \cdots. \end{cases} \tag{3.18}$$

$$\dim H^1(\hat{M}_m, \Theta) = \begin{cases} 0, & m = 0,\ 1. \\ m-1, & m = 2,\ 3, \cdots. \end{cases} \tag{3.19}$$

$$\dim H^2(\hat{M}_m, \Theta) = 0, \qquad m = 0,\ 1, \cdots. \tag{3.20}$$

With these preparations, we consider the complex analytic family $\{M_t : t \in \mathbb{C}\}$ with $M_0 = \hat{M}_m$, $m \geq 2$, defined by (3.11) for k =integral part of $\frac{m}{2}$. For $t \neq 0$, we have by (3.12)

$$M_t = \begin{cases} \hat{M}_0 & \text{if } m \text{ is even.} \\ \hat{M}_1 & \text{if } m \text{ is odd.} \end{cases}$$

Hence by (3.19), we have

$$\text{dim} H^1(M_t, \Theta_t) = \begin{cases} m-1, & t = 0. \\ 0, & t \neq 0. \end{cases}$$

Therefore, by Theorem 3.1.7, the number of moduli $m(\hat{M}_m)$, $m \geq 2$, is not defined even if $H^2(\hat{M}_m, \Theta) = 0$.

Remark :

There exists compact complex manifolds M with $m(M) < \text{dim} H^1(M, \Theta)$.[11] [4]

Chapter 4

Theorem of Completeness

4.1 Theorem of Completeness

Let $(\mathcal{M}, B, \hat{\omega})$=$\{M_t = \hat{\omega}^{-1} : t \in B\}$ be a complex analytic family of compact complex manifolds, where B is a domain of $\mathbb{C}^m$ containing 0.

Theorem 4.1.1 *(**Theorem of Completeness**)[6]*
If $\rho_0 : T_0(B) \to H^1(M_0, \Theta_0)$ is surjective, then the complex analytic family $(\mathcal{M}, B, \hat{\omega})$ is complete at $0 \in B$.

In order to prove the theorem, we must show that for any complex analytic family $(\mathcal{N}, D, \pi)$ with $0 \in D \subset \mathbb{C}^l$ and $\pi^{-1}(0) = M_0$, there exists a subdomain $\Delta \subset D$ containing 0 and a holomorphic map $h : s \to t = h(s)$ with $h(0) = 0$ from Δ into B such that $(\mathcal{N}_\Delta, \Delta, \pi)$ is the complex analytic family induced from $(\mathcal{M}, B, \hat{\omega})$ by h.

Note that if $(\mathcal{N}_\Delta, \Delta, \pi)$ is the complex analytic family induced from $(\mathcal{M}, B, \hat{\omega})$ by h, then for each $s \in \Delta$, we have

$$N_s = \pi^{-1}(s) = M_{h(s)} \times s, \text{ and } \mathcal{N}_\Delta = \bigcup_{s \in \Delta} M_{h(s)} \times s \subset \mathcal{M} \times \Delta$$

is a submanifold of $\mathcal{M} \times \Delta$. Let g be the restriction of the projection $\mathcal{M} \times \Delta \to \mathcal{M}$ to $\mathcal{N}_\Delta$. Then it is clear that g is a holomorphic map of $\mathcal{N}_\Delta$ into $\mathcal{M}$ and maps each N_s biholomorphically onto $M_{h(s)}$. Denoting a point of $N_s = M_{h(s)} \times s$ by (p, s) and identifying $(p, 0) \in N_0 = M_0 \times 0$ with $p \in M_0$, we may consider $\pi^{-1}(0) = N_0 = M_0$. Therefore, if we denote the identity map $(p, 0) \to p$ by g_0, then $g : \mathcal{N}_\Delta \to \mathcal{M}$ is an extension of $g_0 : N_0 \to M_0$. Conversely, we have the following lemma :

Lemma 4.1.2 *If we can extend the identity $g_0 : N_0 = M_0 \to M_0$ to a holomorphic map $g : \mathcal{N}_\Delta \to \mathcal{M}$ such that g maps each N_s, $s \in \Delta$, biholomorphically onto $M_{h(s)}$, then $(\mathcal{N}_\Delta, \Delta, \pi)$ is the complex analytic family induced from $(\mathcal{M}, B, \hat{\omega})$ by h.*

Therefore, in order to prove Theorem4.1.1(Theorem of Completeness), it suffices to show that for any given complex analytic family $(\mathcal{N}, D, \pi)$ with $\pi^{-1}(0) = M_0$, if we take a sufficiently small domain Δ with $0 \in \Delta \subset D$, we can construct a holomorphic map $h : s \to t = h(s), h(0) = 0$, of Δ into B, and a holomorphic map g of $\mathcal{N}_\Delta = \pi^{-1}(\Delta)$ into $\mathcal{M}$ satisfying the following conditions:

(i) g is an extension of the identity $g_0 : \pi^{-1}(0) = M_0 \to M_0$ and

(ii)g maps each $N_s = \pi^{-1}(s)$ biholomorphically onto $M_{h(s)}$.

Proof of Lemma4.1.2: Let $(\hat{\mathcal{N}}, \Delta, \hat{\pi})$ be the complex analytic family induced from $(\mathcal{M}, B, \hat{\omega})$ by $h : \Delta \to B$. We want to show that $(\hat{\mathcal{N}}, \Delta, \hat{\pi})$ and

$(\mathcal{N}_\Delta, \Delta, \pi)$ are biholomorphically equivalent. Since $(\hat{\mathcal{N}}, \Delta, \hat{\pi})$ is induced from $(\mathcal{M}, B, \hat{\omega})$ by h, we have

$$\hat{\pi}^{-1}(s) = M_{h(s)} \times s, \text{ and } \hat{\mathcal{N}} = \bigcup_{s\in\Delta} M_{h(s)} \times s \subset \mathcal{M} \times \Delta$$

is a submanifold of $\mathcal{M} \times \Delta$. Denoting a point of $\mathcal{N}_\Delta$ by q, we consider the holomorphic map

$$\Psi : \mathcal{N}_\Delta \to \mathcal{M} \times \Delta \quad \text{given by}$$

$$\Psi(q) = (g(q), \pi(q)).$$

Since g maps each N_s biholomorphically onto $M_{h(s)}$, Ψ maps N_s biholomorphically onto $\Psi(N_s) = M_{h(s)} \times s$. Hence Ψ maps $\mathcal{N}_\Delta = \bigcup_{s\in\Delta} N_s$ biholomorphically onto $\hat{\mathcal{N}} = \bigcup_{s\in\Delta} M_{h(s)} \times s$. Moreover, it is clear that $\pi = \hat{\pi} \circ \Psi$. Therefore, $(\mathcal{N}_\Delta, \Delta, \pi)$ and $(\hat{\mathcal{N}}, \Delta, \hat{\pi})$ are biholomorphically equivalent. □

4.2 Construction of Formal Power Series of h and g

Let $\{\mathcal{U}_j\}$ be a locally finite open covering of $\mathcal{M}$ and $\{x_j\}$ be the system of local coordinates. Since the problem is local with respect to B, we may assume that

$$B = \{t = (t_1, \ldots, t_m) \in \mathbb{C}^m : |t| = \max_\lambda |t_\lambda| < 1\}$$

and

$$\mathcal{M} = \bigcup_{j=1}^{l} U_j \times B = \bigcup_{j=1}^{l} \mathcal{U}_j,$$

where $U_j \times B = x_j(\mathcal{U}_j) = \mathcal{U}_j$ and $U_j = \{\zeta_j = (\zeta_j^1, \dots, \zeta_j^n) \in \mathbb{C}^n : |\zeta_j| = \max_\alpha |\zeta_j^\alpha| < 1\}$. Here, by abuse of notation, we use $\mathcal{U}_j$ to denote $x_j(\mathcal{U}_j)$. If $\mathcal{U}_j \cap \mathcal{U}_k \neq \emptyset$, then (ζ_j, t) and (ζ_k, t) are the same point of $\mathcal{M}$ if

$$\zeta_j = g_{jk}(\zeta_k, t) = \left(g_{jk}^1(\zeta_k, t), \dots, g_{jk}^n(\zeta_k, t)\right), \tag{4.1}$$

where $g_{jk}^\alpha(\zeta_k, t)$, $\alpha = 1, \dots, n$, are holomorphic functions on $\mathcal{U}_j \cap \mathcal{U}_k$. Similarly, we may assume that

$$D = \{s \in \mathbb{C}^l : |s| < 1\} \text{ and } \mathcal{N} = \bigcup_{j=1}^{l} W_j \times D = \bigcup_{j=1}^{l} \mathcal{W}_j,$$

where $W_j = \{z_j = (z_j^1, \dots, z_j^n) \in \mathbb{C}^n : |z_j| < 1\}$, and (z_j, s) and (z_k, s) are the same point of $\mathcal{N}$ if

$$z_j = f_{jk}(z_k, s) = \left(f_{jk}^1(z_k, s), \dots, f_{jk}^n(z_k, s)\right). \tag{4.2}$$

Moreover, since $N_0 = M_0$, we may also assume that $\mathcal{W}_j \cap N_0 = \mathcal{U}_j \cap M_0$ and the local coordinates $(\zeta_j, 0)$ and $(z_j, 0)$ coincide on $\mathcal{W}_j \cap N_0 = \mathcal{U}_j \cap M_0$. Putting

$$b_{jk}(z_k) = f_{jk}(z_k, 0),$$

by (4.1) and (4.2), we have

$$b_{jk}(\zeta_k) = g_{jk}(\zeta_k, 0).$$

Thus we have

$$N_0 = M_0 = \bigcup_{j=1}^{l} V_j, \quad V_j = W_j \times 0 = \mathcal{W}_j \cap N_0 = \mathcal{U}_j \cap M_0 = U_j \times 0,$$

and the coordinate transformation on $V_j \cap V_k$ is given by

$$z_j^\alpha = b_{jk}^\alpha(z_k), \qquad \alpha = 1, \dots, n.$$

Now, we want to define a holomorphic map $h : s \to t = h(s)$ with $h(0) = 0$ of $\Delta_\varepsilon = \{s \in D : |s| < \varepsilon\}$ into B for a sufficiently small $\varepsilon > 0$, and to extend the identity $g_0 : N_0 \to M_0 = N_0$ to a holomorphic map $g : \pi^{-1}(\Delta_\varepsilon) \to \mathcal{M}$ such that g maps each $N_s = \pi^{-1}(s),\ s \in \Delta_\varepsilon$, into $M_{h(s)} = \hat{\omega}^{-1}(h(s))$, that is, $\hat{\omega} \circ g = h \circ \pi$. Then since g is an extension of the identity g_0, g maps N_s biholomorphically onto $M_{h(s)}$ provided that ε is sufficiently small.

To construct the required h and g, we first try to find out the necessary conditions for which they must satisfy. Suppose that such h and g exist. Then since $g(V_j) = V_j \subset \mathcal{U}_j$ for each $V_j \subset N_0$, $g^{-1}(\mathcal{U}_j) \subset \mathcal{N}_{\Delta_\varepsilon}$ is an open set containing V_j. Therefore, we can choose a sufficiently small $\varepsilon > 0$ such that for $j = 1, \ldots, l$, the subdomain

$$\mathcal{W}_j^* = \{(z_j, s)\ :\ |z_j| < 1,\ |s| < \varepsilon\}$$

of $\mathcal{W}_j$ is contained in $g^{-1}(\mathcal{U}_j)$:

$$V_j \subset \mathcal{W}_j^* \subset g^{-1}(\mathcal{U}_j) \cap \mathcal{W}_j.$$

We put

$$g(z_j, s) = (\zeta_j, t) = (g_j(z_j, s), h(s)) = \big(g_j^1(z_j, s), \ldots, g_j^n(z_j, s), h(s)\big) \tag{4.3}$$

on $\mathcal{W}_j^*$. Then each g_j^α is holomorphic function of $z_j^1, \ldots, z_j^n, s_1, \ldots, s_l$ defined on $\mathcal{W}_j^*$. Moreover, since $g(z_j, 0) = (\zeta_j, 0) = (z_j, 0)$, we have

$$g_j(z_j, 0) = z_j \qquad \text{and} \qquad h(0) = 0. \tag{4.4}$$

Expanding $g_j(z_j, s)$ and $h(s)$ into power series of $s_1, \ldots, s_l$, we get

$$g_j(z_j, s) = z_j + \sum_{\nu=1}^{\infty} g_{j|\nu}(z_j, s) \tag{4.5}$$

and

$$h(s)=\sum_{\nu=1}^{\infty} h_\nu(s), \qquad h_\nu(s)=(h_{1|\nu}(s),\dots,h_{l|\nu}(s)), \tag{4.6}$$

where $g_{j|\nu}(z_j,s)=\sum_{\nu_1+\cdots+\nu_l=\nu} g_{j\nu_1\cdots\nu_l}(z_j)s_1^{\nu_1}\cdots s_l^{\nu_l}$ and each component $g^{\alpha}_{j\nu_1\cdots\nu_l}(z_j)$ of the coefficient

$$g_{j\nu_1\cdots\nu_l}(z_j)=\left(g^1_{j\nu_1\cdots\nu_l}(z_j),\dots,g^n_{j\nu_1\cdots\nu_l}(z_j)\right)$$

is a holomorphic function of $z_j^1,\dots,z_j^n$ defined on V_j. Note that the power series $g_j(z_j,s)$ of $s_1,\dots,s_l$ converge for $|s|<\varepsilon$. Moreover, there is a basic relation

$$g_j\left(f_{jk}(z_k,s),s\right)=g_{jk}\left(g_k(z_k,s),h(s)\right) \tag{4.7}$$

on $\mathcal{W}_j^*\cap\mathcal{W}_k^*$.

Now, we try to construct the formal power series $g_j(z_j,s)$ and $h(s)$ on each V_j such that (4.7) holds. As before, we reduce (4.7) into the following system of infinitely many congruences :

$$\mathbf{P}(\nu) \quad \cdots\cdots \quad g_j^{\nu}\left(f_{jk}(z_k,s),s\right)) \overset{\nu}{\equiv} g_{jk}\left(g_k^{\nu}(z_k,s),h^{\nu}(s)\right),\ \nu=0,1,2,\cdots$$

As usual, we want to construct $g_j^{\nu}(z_j,s)$ and $h^{\nu}(s)$ by induction on ν so that $\mathbf{P}(\nu)$ holds on $V_j\cap V_k\neq\emptyset$. It is clear that $\mathbf{P}(\nu)$ holds for $\nu=0$. Suppose that $g_j^{\nu-1}(z_j,s)$ and $h^{\nu-1}(s)$ are already constructed in such a way that

$$g_j^{\nu-1}\left(f_{jk}(z_k,s),s\right) \overset{\nu-1}{\equiv} g_{jk}\left(g_k^{\nu-1}(z_j,s),h^{\nu-1}(s)\right) \tag{4.8}$$

holds on each $V_j \cap V_k \neq \emptyset$. Next, we consider $\mathbf{P}(\nu)$. By an elementary calculation, one can show that $\mathbf{P}(\nu)$ is equivalent to the following congruence:

$$\begin{aligned} & g_j^{\nu-1}\left(f_{jk}(z_k,s),s\right) - g_{jk}\left(g_k^{\nu-1}(z_k,s),h^{\nu-1}(s)\right) \\ \overset{\nu}{\equiv} \; & \sum_\beta \frac{\partial z_j}{\partial z_k^\beta}\cdot g_{k|\nu}^\beta(z_k,s) - g_{j|\nu}(z_j,s) + \sum_r \left(\frac{\partial g_{jk}(z_k,t)}{\partial t_r}\right)_{t=0}\cdot h_{r|\nu}(s), \end{aligned} \tag{4.9}$$

where $z_j = f_{jk}(z_k,0)$ and $h(s) = t$. By the induction hypothesis, the left hand side of $(4.9) \overset{\nu-1}{\equiv} 0$. Thus if we put

$$\Gamma_{jk|\nu}(z_j,s) = \left(g_j^{\nu-1}\left(f_{jk}(z_k,s),s\right) - g_{jk}(g_k^{\nu-1}(z_k,s),h^{\nu-1}(s))\right)_\nu, \tag{4.10}$$

then we have

$$\Gamma_{jk|\nu}(z_j,s) \overset{\nu}{\equiv} g_j^{\nu-1}\left(f_{jk}(z_k,s),s\right) - g_{jk}\left(g_k^{\nu-1}(z_k,s),h^{\nu-1}(s)\right). \tag{4.11}$$

Therefore, $\mathbf{P}(\nu)$ is equivalent to the following equality :

$$\begin{aligned} & \Gamma_{jk|\nu}(z_j,s) \\ = \; & \sum_{r=1}^m \left(\frac{\partial g_{jk}(z_k,t)}{\partial t_r}\right)_{t=0}\cdot h_{r|\nu}(s) + \sum_{\beta=1}^m \frac{\partial z_j}{\partial z_k^\beta}\cdot g_{k|\nu}^\beta(z_k,s) - g_{j|\nu}(z_j,s), \end{aligned} \tag{4.12}$$

where $z_j = b_{jk}(z_k)$.

Put

$$\begin{aligned} \theta_{rjk} &= \sum_{\alpha=1}^n \left(\frac{\partial g_{jk}^\alpha(z_k,t)}{\partial t_r}\right)_{t=0} \frac{\partial}{\partial z_j^\alpha}, \\ \Gamma_{jk|\nu}(s) &= \sum_{\alpha=1}^n \Gamma_{jk|\nu}^\alpha(z_j,s)\frac{\partial}{\partial z_j^\alpha}, \\ g_{k|\nu}(s) &= \sum_{\beta=1}^n g_{k|\nu}^\beta(z_k,s)\frac{\partial}{\partial z_k^\beta}, \end{aligned}$$

where $z_j = b_{jk}(z_k)$. Then in terms of the above notation, $\mathbf{P}(\nu)$ is written as

$$\Gamma_{jk|\nu}(s) = \sum_{\alpha=1}^m h_{r|\nu}(s)\theta_{rjk} + g_{k|\nu}(s) - g_{j|\nu}(s). \tag{4.13}$$

Note that the 1−cocycle $\{\theta_{rjk}\} \in Z^1(\mathcal{V}, \Theta_0)$ represents the infinitesimal deformation $\theta_r = \rho_0\left(\frac{\partial}{\partial t_r}\right) \in H^1(M_0, \Theta_0)$, where $\mathcal{V} = \{V_j\}$ is a finite open covering of $M_0 = N_0$. Moreover, for any fixed $s \in D$, we have the following lemma :

Lemma 4.2.1 $\{\Gamma_{jk|\nu}(s)\} \in Z^1(\mathcal{V}, \Theta_0)$.

Proof : It suffices to prove that

$$\Gamma_{jk|\nu}(s) - \Gamma_{ik|\nu}(s) + \Gamma_{ij|\nu}(s) = 0. \tag{4.14}$$

Using matrix notation, (4.14) is written in the form :

$$\Gamma_{ik|\nu}(z_i, s) = \Gamma_{ij|\nu}(z_i, s) + A_{ij}(z_j)\Gamma_{jk|\nu}(z_j, s), \tag{4.15}$$

where $z_i = b_{ij}(z_j)$ and $A_{ij}(z_j) = \left(\frac{\partial z_i^\alpha}{\partial z_j^\beta}\right)_{\alpha,\beta=1,\cdots,n}$. For simplicity, we denote $f_{jk}(z_k, s)$ and $g_k^{\nu-1}(z_k, s)$ by $f_{jk}(s)$ and $g_k^{\nu-1}(s)$ respectively. Then by (4.11), we have

$$\Gamma_{ik|\nu}(z_i, s) \overset{\nu}{\equiv} g_i^{\nu-1}(f_{ik}(s), s) - g_{ik}\left(g_k^{\nu-1}(s), h^{\nu-1}(s)\right), \tag{4.16}$$

and

$$\Gamma_{jk|\nu}(z_j, s) \overset{\nu}{\equiv} g_j^{\nu-1}(f_{jk}(s), s) - g_{jk}\left(g_k^{\nu-1}(s), h^{\nu-1}(s)\right). \tag{4.17}$$

Then by (4.17), we have

$$\begin{aligned}
& g_{ik}\left(g_k^{\nu-1}(s), h^{\nu-1}(s)\right) \\
= \;& g_{ij}\left(g_{jk}(g_k^{\nu-1}(s), h^{\nu-1}(s)), h^{\nu-1}(s)\right) \\
\overset{\nu}{\equiv} \;& g_{ij}\left(g_j^{\nu-1}(f_{jk}(s), s) - \Gamma_{jk|\nu}(z_j, s), h^{\nu-1}(s)\right) \\
\overset{\nu}{\equiv} \;& g_{ij}\left(g_j^{\nu-1}(f_{jk}(s), s), h^{\nu-1}(s)\right) - \sum_{\beta=1}^{n} \frac{\partial g_{ij}}{\partial \zeta_j^\beta}\left(g_j^{\nu-1}(f_{jk}(s), s), h^{\nu-1}(s)\right)\Gamma_{jk|\nu}^\beta(z_j, s) \\
\overset{\nu}{\equiv} \;& g_{ij}\left(g_j^{\nu-1}(f_{jk}(s), s), h^{\nu-1}(s)\right) - \sum_{\beta=1}^{n} \frac{\partial z_i}{\partial z_j^\beta}\Gamma_{jk|\nu}^\beta(z_j, s),
\end{aligned} \tag{4.18}$$

where $\frac{\partial z_i}{\partial z_j^\beta} = \frac{\partial b_{ij}}{\partial z_j^\beta}$. Then by putting (4.18) into (4.16), we get

$$\begin{aligned} & \Gamma_{ik|\nu}(z_i, s) \\ \stackrel{\nu}{\equiv} \ & g_i^{\nu-1}\left(f_{ik}(s), s\right) - g_{ij}\left(g_j^{\nu-1}(f_{jk}(s), s), h^{\nu-1}(s)\right) + A_{ij}(z_j)\Gamma_{jk|\nu}(z_j, s). \end{aligned} \tag{4.19}$$

But, by (4.11), we have

$$\begin{aligned} & g_i^{\nu-1}\left(f_{ik}(s), s\right) - g_{ij}\left(g_j^{\nu-1}(f_{jk}(s), s), h^{\nu-1}(s)\right) \\ \stackrel{\nu}{\equiv} \ & g_i^{\nu-1}\left(f_{ij}(f_{jk}(s), s), s\right) - g_{ij}\left(g_j^{\nu-1}(f_{jk}(s), s), h^{\nu-1}(s)\right) \\ \stackrel{\nu}{\equiv} \ & \Gamma_{ij|\nu}\left(b_{ij}(f_{jk}(s)), s\right) \\ \stackrel{\nu}{\equiv} \ & \Gamma_{ij|\nu}(z_i, s). \end{aligned}$$

Therefore, by substituting it into (4.19), we get (4.15). □

Thus, $\mathbf{P}(\nu)$ may be written in the following form :

$$\{\Gamma_{jk|\nu}(s)\} = \sum_{r=1}^{m} h_{r|\nu}(s)\{\theta_{rjk}\} + \delta\{g_{j|\nu}(s)\}, \tag{4.20}$$

where $\{\Gamma_{jk|\nu}(s)\}$, $\{\theta_{rjk}\} \in Z^1(\mathcal{V}, \Theta_0)$ and $\{g_{j|\nu}(s)\} \in C^0(\mathcal{V}, \Theta_0)$.

Lemma 4.2.2 *For any $\{\Gamma_{jk}\} \in Z^1(\mathcal{V}, \Theta_0)$, there exists $h_r \in \mathbb{C}$, $r = 1, \ldots, m$, and $\{g_j\} \in C^0(\mathcal{V}, \Theta_0)$ such that*

$$\{\Gamma_{jk}\} = \sum_{r=1}^{m} h_r\{\theta_{rjk}\} + \delta\{g_j\}. \tag{4.21}$$

Proof : Let $\gamma \in H^1(M_0, \Theta_0)$ be the cohomology class of the 1−cocycle $\{\Gamma_{jk}\}$. Since by assumption, $\rho_0 : T_0(B) \to H^1(M_0, \Theta_0)$ is surjective, there exists $h_r \in \mathbb{C}$, $r = 1, \ldots, m$, such that

$$\gamma = \sum_{r=1}^{m} h_r\theta_r.$$

Therefore, the cohomology class of $\{\Gamma_{jk}\} - \sum_{r=1}^{m} h_r\{\theta_{rjk}\}$ is 0. As $H^1(\mathcal{V}, \Theta_0) \hookrightarrow H^1(M_0, \Theta_0)$, we have $\{\Gamma_{jk}\} - \sum_{r=1}^{m} h_r\{\theta_{rjk}\} \in \delta C^0(\mathcal{V}, \Theta_0)$. $\square$

Therefore, the formal power series $h(s)$ and $g_j(z_j, s)$ can be constructed so that $\mathbf{P}(\nu)$ holds for all $\nu \in \mathbb{N}$. In next section, we will show that if $h_{r|\nu}(s)$ and $\{g_{j|\nu}(z_j, s)\}$ are chosen appropriately, then $h(s)$ and $g_j(z_j, s)$ converge absolutely in $|s| < \varepsilon$ provided that ε is sufficiently small.

4.3 Proof of Convergence

Note that in vector notation, (4.21) is written in the following form :

$$\Gamma_{jk}(z_j) = \sum_{r=1}^{m} h_r \theta_{rjk}(z_j) + A_{jk}(z_k) g_k(z_k) - g_j(z_j), \tag{4.22}$$

where $A_{jk}(z_k) = \left(\frac{\partial z_j^\alpha}{\partial z_k^\beta}\right)_{\alpha,\beta=1,\cdots,n.}$. Since each $V_j = \{z_j \in \mathbb{C}^n \ : \ |z_j| < 1\}$ is a coordinate polydisk, we may assume that the coordinate function z_j is defined on a domain of N_0 containing $\overline{V_j}$. Therefore, there exists a constant C_1 such that

$$\frac{\partial z_j^\alpha}{\partial z_k^\beta} < C_1, \text{ for all } \alpha, \beta = 1, \ldots, n, \text{ and } j, k = 1, \ldots, l.$$

Definition 4.3.1 *We define the norm of the matrix $A_{jk}(z_k)$ by*

$$|A_{jk}(z_k)| = \sup_{\zeta} \frac{|A_{jk}(z_k)\zeta|}{|\zeta|}, \qquad \zeta \in \mathbb{C}^n, \ \zeta \neq 0. \tag{4.23}$$

Then it is clear that there exists a constant C_2 such that for all $j, k = 1, \ldots, l$, we have

$$|A_{jk}(z_k)| < C_2 \quad \text{on} \quad V_j \cap V_k \neq \emptyset. \tag{4.24}$$

Definition 4.3.2 *We define the norm of a* $1-$*cocycle* $|\Gamma| = \{\Gamma_{jk}\} \in Z^1(\mathcal{V}, \Theta_0)$ *by*

$$|\Gamma| = \max_{j,k} \sup_{z_j \in V_j \cap V_k} |\Gamma_{jk}(z_j)|, \quad \Gamma_{jk}(z_j) = (\Gamma^1_{jk}(z_j), \cdots, \Gamma^n_{jk}(z_j)). \tag{4.25}$$

Moreover, if $|\Gamma| < \infty$, *we define* $\imath(\Gamma)$ *by*

$$\imath(\Gamma) = \inf \max_{r,j} \left\{ |h_r|, \sup_{z_j \in U_j} |g_j(z_j)| \right\}, \tag{4.26}$$

where inf *is taken with respect to all the solutions* h_r, $r = 1, \ldots, m$, $g_j(z_j)$ *of* (4.21).

Lemma 4.3.3 *For a* $1-$*cocycle* $\Gamma = \{\Gamma_{jk}\}$ *with* $|\Gamma| < \infty$, *if* h_r, $r = 1, \ldots, m$, $\{g_j\}$ *is a solution of* (4.22), *then* $|g_j(z_j)| = |(g_j^1(z_j), \cdots, g_j^n(z_j))|$ *is bounded on* V_j, $1 \le j \le l$.

Proof : Put

$$V_j^\delta = \{z_j \in V_j \ : \ |z_j| < 1 - \delta\}.$$

Then, since $\{V_j\}$ is a finite open covering of N_0 and N_0 is compact, we have

$$N_0 = \bigcup_{j=1}^{l} V_j^\delta$$

for a sufficiently small δ. As $g_j(z_j)$ is holomorphic on V_j, it is bounded on $V_j^\delta \subset \overline{V_j^\delta} \subset V_j$. If $z_j \notin V_j^\delta$, then $z_j \in V_k^\delta$ for some $k \neq j$. Therefore, by (4.22), we have the following equality on $V_j \cap V_k^\delta$:

$$g_j(z_j) = \sum_{r=1}^{m} h_r \theta_{rjk}(z_j) + A_{jk}(z_k) g_k(z_k) - \Gamma_{jk}(z_j), \tag{4.27}$$

where $z_j = b_{jk},\ z_k \in V_k^\delta$. Since $\theta_{rjk}^\alpha(z_j) = \left(\frac{\partial g_{jk}^\alpha(z_k,t)}{\partial t_r}\right)_{t=0}$ are bounded holomorphic functions, there exists a constant C_3 such that for all $j, k = 1, \ldots, l$, we have the following inequality :

$$|\theta_{rjk}(z_j)| = |(\theta_{rjk}^1(z_j), \cdots, \theta_{rjk}^n(z_j))| < C_3 \text{ on } V_j \cap V_K \neq \emptyset. \tag{4.28}$$

Moreover, since $|A_{jk}(z_k)| < C_2$ by (4.24) and $|\Gamma| < \infty$ by assumption, and since $g_k(z_k)$ is bounded on V_k^δ, $|g_j(z_j)|$ is bounded on $V_j \cap V_k^\delta$ by (4.27). As V_j is covered by V_j^δ and a finite number of $V_k^\delta,\ k \neq j$, it follows that $|g_j(z_j)|$ is bounded on $V_j,\ j = 1, \ldots, l$. □

Lemma 4.3.4 *For any $\Gamma = \{\Gamma_{jk}\} \in Z^1(\mathcal{V}, \Theta_0)$ with $|\Gamma| < \infty$, there exists solutions $h_r,\ r = 1, \ldots, m,\ \{g_j\}$ of (4.21) such that the following inequalities hold.*

$$|h_r| \leq C_4|\Gamma|, \qquad |g_j(z_j)| \leq C_4|\Gamma|, \tag{4.29}$$

where C_4 is a constant independent of $\Gamma = \{\Gamma_{jk}\}$.

Proof : It suffices to prove that

$$\imath(\Gamma) \leq C|\Gamma| \tag{4.30}$$

for some constant C independent of Γ. Suppose that (4.30) does not hold. Then for any $n \in \mathbb{N}$, there exists $\Gamma^{(n)} \in Z^1(\mathcal{V}, \Theta_0)$ such that

$$\imath(\Gamma^{(n)}) > n|\Gamma^{(n)}|.$$

Replacing $\Gamma^{(n)}$ by $\frac{\Gamma^{(n)}}{\imath(\Gamma^{(n)})}$, we obtain a sequence of 1−cocycles $\Gamma^{(n)}=\{\Gamma^{(n)}_{jk}\} \in Z^1(\mathcal{V}, \Theta_0)$ such that

$$\imath(\Gamma^{(n)}) = 1, \qquad |\Gamma^{(n)}| < \frac{1}{n}. \tag{4.31}$$

The equality $\imath(\Gamma^{(n)}) = 1$ implies that there exists a solution $h^{(n)}_r$, $r = 1, \ldots, m$, $\{g^{(n)}_j\}$ of (4.21) for $\Gamma = \Gamma^{(n)}$ such that

$$|h^{(n)}_r| < 5, \qquad |g^{(n)}_j(z_j)| < 5.$$

Therefore, we may assume that the sequences $\{h^{(n)}_r\}$, $r = 1, \ldots, m$, converge and $\{g^{(n)}_j(z_j)\}$ converges uniformly on each compact subset of V_j. Since $\overline{V^\delta_j} \subset V_j$ is compact, $\{g^{(n)}_j(z_j)\}$ converges uniformly on $V^\delta_j \subset \overline{V^\delta_j}$. If $z_j \notin V^\delta_j$, then $z_j \in V^\delta_k$ for some $k \neq j$. Hence by (4.22), we have the following equality on $V_j \cap V^\delta_k$:

$$g^{(n)}_j(z_j) = \sum_{r=1}^{m} h^{(n)}_r \theta_{rjk}(z_j) + A_{jk}(z_k) g^{(n)}_k(z_k) - \Gamma^{(n)}_{jk}(z_j),$$

where $z_j = b_{jk}(z_k)$, $z_k \in V^\delta_k$. Since $|\Gamma^{(n)}_{jk}(z_j)| \leq |\Gamma^{(n)}| < \frac{1}{n} \to 0$ as $n \to \infty$, it follows from the previous results that the right hand side of the above equality converges uniformly on $V_j \cap V^\delta_k$. Since $V_j \subset \cup^l_{k=1} V^\delta_k = N_0$, we conclude that $\{g^{(n)}_j(z_j)\}$ converges uniformly on V_j. Putting

$$h_r = \lim_{n\to\infty} h^{(n)}_r \quad \text{and} \quad g_j(z_j) = \lim_{n\to\infty} g^{(n)}_j(z_j),$$

as $|\Gamma^{(n)}_{jk}(z_j)| \leq |\Gamma^{(n)}| < \frac{1}{n} \to 0$ when $n \to \infty$, we have

$$0 = \sum_{r=1}^{m} h_r \theta_{rjk}(z_j) + A_{jk}(z_k) g_k(z_k) - g_j(z_j).$$

Therefore, by putting

$$\hat{h}_r^{(n)} = h_r^{(n)} - h_r \quad \text{and} \quad \hat{g}_j^{(n)}(z_j) = g_j^{(n)}(z_j) - g_j(z_j),$$

we get

$$\Gamma_{jk}^{(n)}(z_j) = \sum_{r=1}^{m} \hat{h}_r^{(n)} \theta_{rjk}(z_j) + A_{jk}(z_k)\hat{g}_k^{(n)} - \hat{g}_j^{(n)}(z_j).$$

This implies that $\hat{h}_r^{(n)}$, $r = 1, \ldots, m$, and $\{\hat{g}_j^{(n)}\}$ is also a solution of (4.21) for $\Gamma = \Gamma^{(n)}$. But we have $|\hat{h}_r^{(n)}| \to 0$ and $\sup_{z_j \in V_j} |\hat{g}_j^{(n)}(z_j)| \to 0$ as $n \to \infty$, which contradicts the fact that $\imath(\Gamma) = 1$. □

Now, using the method of majorant series as in section 2.5, we want to show that

$$h(s) \ll A(s) \quad \text{and} \quad g_j(z_j, s) - z_j \ll A(s), \tag{4.32}$$

where $A(s) = \frac{b}{16c} \sum_{\nu=1}^{\infty} \frac{c^\nu (s_1 + \cdots + s_l)^\nu}{\nu^2}$, and $b, c > 0$ are properly chosen. Recall that

$$A(s)^2 \ll \frac{b}{c} A(s).$$

Moreover, using mathematical induction, one can show that

$$A(s)^\nu \ll \left(\frac{b}{c}\right)^{\nu-1} A(s), \quad \nu = 2, 3, \ldots \tag{4.33}$$

To prove (4.32), it suffices to show that

$$\mathbf{T}(\nu) \quad \cdots\cdots \quad h^\nu(s) \ll A(s) \quad \text{and} \quad g_j^\nu(z_j, s) - z_j \ll A(s) \text{ for } \nu \in \mathbb{N}.$$

As before, we prove $\mathbf{T}(\nu)$ by induction on ν. Since

$$A_1(s) = \frac{b}{16}(s_1 + \cdots + s_l),$$

$\mathbf{T}(\nu)$ is obviously true if b is large enough. Suppose that $\mathbf{T}(\nu)$ is true. Next, we want to show that

$$\Gamma_{jk|\nu}(s) \ll KA(s),$$

where K is a positive constant, and then by (4.29) to conclude that

$$h_{r|\nu}(s) \ll C_4 KA(s) \quad \text{and} \quad g_{j|\nu}(s) \ll C_4 KA(s).$$

By (4.10), we have

$$\Gamma_{jk|\nu}(z_j, s) = \left(g_j^{\nu-1}(f_{jk}(z_k, s), s)\right)_\nu - \left(g_{jk}(g_k^{\nu-1}(z_k, s), h^{\nu-1}(s))\right)_\nu . \quad (4.34)$$

We first estimate the first term of the right hand side of (4.34) :

$$\left(g_j^{\nu-1}(f_{jk}(z_k, s), s)\right)_\nu .$$

Since $f_{jk}(z_k, s)$ are vector-valued holomorphic functions, we may assume that

$$f_{jk}(z_k, s) - b_{jk}(z_k) \ll \hat{A}(s) = \frac{\hat{b}}{16\hat{c}} \sum_{\nu=1}^{\infty} \frac{\hat{c}^\nu (s_1 + \cdots + s_l)^\nu}{\nu^2}, \quad \hat{b}, \hat{c} > 0, \quad (4.35)$$

on $V_j \cap V_k$. Put

$$G(z_j, s) = g_j^{\nu-1}(z_j, s) - z_j.$$

Then, by induction hypothesis, we have

$$\sum_{\nu_1 + \cdots + \nu_l \geq 1} G_{\nu_1 \cdots \nu_l}(z_j) s_1^{\nu_1} \cdots s_l^{\nu_l} = G(z_j, s) \ll A(s) = \sum_{\nu_1 + \cdots + \nu_l \geq 1} A_{\nu_1 \cdots \nu_l} s_1^{\nu_1} \cdots s_l^{\nu_l}.$$

That is

$$|G_{\nu_1 \cdots \nu_l}(z_j)| \leq A_{\nu_1 \cdots \nu_l}, \qquad z_j \in U_j. \quad (4.36)$$

Let $z_j \in V_j^\delta$, since $V_j^\delta = \{z_j \in V_j : |z_j| < 1 - \delta\}$, $G_{\nu_1 \cdots \nu_l}(z_j + \zeta)$ is a vector-valued holomorphic function of $\zeta = (\zeta_1, \ldots, \zeta_n)$, $|\zeta| < \delta$. Expanding $G_{\nu_1 \cdots \nu_l}(z_j + \zeta)$ into power series in $\zeta_1, \ldots, \zeta_n$, we get

$$G_{\nu_1 \cdots \nu_l}(z_j + \zeta) = \sum_{\mu_1, \cdots, \mu_n} G_{\nu_1 \cdots \nu_l \mu_1 \cdots \mu_n}(z_j) \zeta_1^{\mu_1} \cdots \zeta_n^{\mu_n}.$$

By Cauchy's integral formula and (4.36), we get

$$|G_{\nu_1 \cdots \nu_l \mu_1 \cdots \mu_n}(z_j)| \leq \frac{A_{\nu_1 \cdots \nu_l}}{\delta^{\mu_1 + \cdots + \mu_n}}.$$

Therefore, we obtain

$$G_{\nu_1 \cdots \nu_l}(z_j + \zeta) - G_{\nu_1 \cdots \nu_l}(z_j) \ll \sum_{\mu_1 + \cdots + \mu_n \geq 1} \frac{A_{\nu_1 \cdots \nu_l}}{\delta^{\mu_1 + \cdots \mu_n}} \zeta_1^{\mu_1} \cdots \zeta_n^{\mu_n}.$$

This implies that

$$G(z_j + \zeta, s) - G(z_j, s) \ll A(s) \sum_{\mu_1 + \cdots + \mu_n \geq 1} \frac{\zeta_1^{\mu_1} \cdots \zeta_n^{\mu_n}}{\delta^{\mu_1 + \cdots + \mu_n}}.$$

If we put $\zeta = f_{jk}(z_k, s) - z_j$, then by (4.35), we have

$$\begin{aligned} G\left(f_{jk}(z_k, s), s\right) - G(z_j, s) &\ll A(s) \sum_{\mu_1 + \cdots + \mu_n \geq 1} \frac{\hat{A}(s)^{\mu_1 + \cdots + \mu_n}}{\delta^{\mu_1 + \cdots + \mu_n}} \\ &= A(s) \left\{ \left[\sum_{\mu=0}^{\infty} \left(\frac{\hat{A}(s)}{\delta} \right)^{\mu} \right]^n - 1 \right\}. \end{aligned} \tag{4.37}$$

Taking a sufficiently large $\hat{c}$, we may assume that

$$\frac{\hat{b}}{\hat{c}\delta} < \frac{1}{2}. \tag{4.38}$$

Then by (4.33), we have

$$\begin{aligned}
\left[\sum_{\mu=0}^{\infty}\left(\frac{\hat{A}(s)}{\delta}\right)^{\mu}\right]^{n}-1 &\ll \left[1+\sum_{\mu=1}^{\infty}\frac{\hat{A}(s)^{\mu}}{\delta^{\mu}}\right]^{n}-1\\
&\ll \left[1+\frac{1}{\delta}\sum_{\mu=1}^{\infty}\left(\frac{\hat{b}}{\hat{c}\delta}\right)^{\mu-1}\hat{A}(s)\right]^{n}-1\\
&\ll \left[1+\frac{2}{\delta}\hat{A}(s)\right]^{n}-1\\
&\ll \sum_{k=1}^{n}\binom{n}{k}\left(\frac{2}{\delta}\right)^{k}\hat{A}(s)^{k}\\
&\ll \frac{2}{\delta}\sum_{k=1}^{n}\binom{n}{k}\left(\frac{2\hat{b}}{\hat{c}\delta}\right)^{k-1}\hat{A}(s)\\
&\ll \frac{2}{\delta}\sum_{k=0}^{n}\binom{n}{k}\hat{A}(s)\\
&\ll \frac{2^{n+1}}{\delta}\hat{A}(s).
\end{aligned}$$

Putting this result into (4.37), we get

$$G\left(f_{jk}(z_k,s),s\right)-G(z_j,s)\ll\frac{2^{n+1}}{\delta}A(s)\hat{A}(s). \tag{4.39}$$

Note that if we take $b,c>0$ such that $b>\hat{b}$ and $c>\hat{c}$, then we have

$$\hat{A}(s)\ll\frac{\hat{b}}{b}A(s).$$

Using this fact together with (4.33), (4.39) is reduced to

$$\begin{aligned}
G\left(f_{jk}(z_k,s),s\right)-G(z_j,s) &\ll \frac{2^{n+1}}{\delta}\frac{\hat{b}}{b}A(s)^2\\
&\ll \frac{2^{n+1}\hat{b}}{c\delta}A(s).
\end{aligned}$$

As

$$G(z_j,s)=g_j^{\nu-1}(z_j,s)-z_j \quad\text{and}\quad f_{jk}(z_k,s)-z_j\ll\hat{A}(s)\ll\frac{\hat{b}}{b}A(s),$$

we have

$$g_j^{\nu-1}\left(f_{jk}(z_k,s),s\right)-g_j^{\nu-1}(z_j,s)\ll\left(\frac{2^{n+1}\hat{b}}{c\delta}+\frac{\hat{b}}{b}\right)A(s).$$

This implies that

$$\left(g_j^{\nu-1}\left(f_{jk}(z_k,s),s\right)\right)_\nu \ll \left(\frac{2^{n+1}\hat{b}}{c\delta}+\frac{\hat{b}}{b}\right)A(s). \tag{4.40}$$

Next, we want to estimate the second term of the right hand side of (4.34) :

$$\left(g_{jk}(g_k^{\nu-1}(z_k,s),h^{\nu-1}(s))\right)_\nu.$$

Expanding $g_{jk}(z_k+\zeta,t)$ into power series in $\zeta_1, \ldots, \zeta_n, t_1, \ldots, t_m$, since $g_{jk}(z_k,t)$ are vector-valued holomorphic functions, we may assume that

$$\begin{aligned} & g_{jk}(z_k+\zeta,t) - \text{constant term} - \text{linear term} \\ \ll & \sum_{\mu=2}^{\infty} \hat{a}^\mu(\zeta_1+\cdots+\zeta_n+t_1+\cdots+t_m)^\mu, \qquad \hat{a}>0. \end{aligned}$$

If we put $\zeta = g_k^{\nu-1}(z_k,s)-z_k)-z_k$ and $t=h^{\nu-1}(s)$, then since $\zeta \ll A(s)$ and $t \ll A(s)$ by induction hypothesis, we get

$$\begin{aligned} & g_{jk}\left(g_k^{\nu-1}(z_k,s),h^{\nu-1}(s)\right) - \text{constant term} - \text{linear term} \\ \ll & \sum_{\mu=2}^{\infty} \hat{a}^\mu(n+m)^\mu A(s)^\mu \\ \ll & \sum_{\mu=2}^{\infty} \hat{a}^\mu(m+n)^\mu \left(\tfrac{b}{c}\right)^{\mu-1} A(s). \qquad (\text{ by (4.33) }) \end{aligned}$$

This implies that

$$\left(g_{jk}(g_k^{\nu-1}(z_k,s),h^{\nu-1}(s))\right)_\nu \ll \frac{b\hat{a}^2(m+n)^2}{c}\sum_{\mu=0}^{\infty}\left(\frac{b\hat{a}(m+n)}{c}\right)^\mu A(s).$$

Taking a constant c such that $\frac{b\hat{a}(m+n)}{c}<\frac{1}{2}$, we have

$$\left(g_{jk}(g_k^{\nu-1}(z_k,s),h^{\nu-1}(s))\right)_\nu \ll \frac{2b\hat{a}^2(m+n)^2}{c}A(s). \tag{4.41}$$

Combining (4.40) and (4.41), we get

$$\Gamma_{jk|\nu}(z_j,s) \ll C^*A(s) = \left(\frac{2^{n+1}\hat{b}}{c\delta}+\frac{\hat{b}}{b}+\frac{2b\hat{a}^2(m+n)^2}{c}\right)A(s) \tag{4.42}$$

on $V_j^\delta \cap V_k$. Since

$$\Gamma_{jk|\nu}(z_j, s) = A_{ji}(z_i)\Gamma_{ik|\nu}(z_i, s) - A_{ji}(z_i)\Gamma_{ij|\nu}(z_i, s) \quad \text{and} \quad |A_{ji}(z_i)| < C_2,$$

it is easy to see that

$$\Gamma_{jk|\nu}(z_j, s) \ll 2C_2C^*A(s) \qquad \text{on } V_j \cap V_k.$$

Therefore, it follows from Lemma 4.3.4 that

$$h_{r|\nu}(s) \ll C_4 2C_2C^*A(s),\ r = 1, \ldots, m, \qquad g_{j|\nu}(s) \ll C_4 2C_2C^*A(s).$$

Note that $C_2C_4C^*$ is independent of ν. Moreover, it is clear that we may choose b and c such that

$$2C_2C_4C^* \leq 1.$$

Consequently, putting

$$h^\nu(s) = h^{\nu-1}(s) + h_\nu(s) \quad \text{and} \quad g_j^\nu(z_j, s) = g_j^{\nu-1}(z_j, s) + g_{j|\nu}(z_j, s),$$

we get the result :

$$h^\nu(s) \ll A(s) \quad \text{and} \quad g_j^\nu(z_j, s) - z_j \ll A(s).$$

As the radius of convergence of the power series $\sum_{\nu=1}^{\infty} \frac{s^\nu}{\nu^2}$ is 1, $h(s)$ converges for $|s| < \frac{1}{lc}$, and $g_j(z_j, s)$ converges absolutely and uniformly on V_j for $|s| < \frac{1}{lc}$.

With these preparations, we prove the theorem of completeness as follows:

Recall that the complex analytic families $(\mathcal{N}, D, \pi)$ and $(\mathcal{M}, B, \hat{\omega})$ are represented as below :

$$\mathcal{N} = \bigcup_{j=1}^{l} V_j \times D = \bigcup_{j=1}^{l} \mathcal{W}_j, \quad \mathcal{M} = \bigcup_{j=1}^{l} V_j \times B = \bigcup_{j=1}^{l} \mathcal{U}_j,$$

where $V_j = \{z_j \in \mathbb{C}^n : |z_j| < 1\}$, $D = \{s \in \mathbb{C}^l : |s| < 1\}$ and $B = \{t \in \mathbb{C}^m : |t| < 1\}$. By the previous results, there exists holomorphic maps :

$$G_j : (z_j, s) \to (\zeta_j, t) = (g_j(z_j, s), h(s)), \; G_j(z_j, o) = (\zeta_j, 0) = (z_j, 0), \quad (4.43)$$

from $V_j \times \Delta \subset V_j \times D$ into $\mathbb{C}^n \times \mathbb{C}^m$ provided that Δ is sufficiently small. Now we want to show that these G_j define a holomorphic map

$$g : \mathcal{N}_\Delta = \pi^{-1}(\Delta) \to \mathcal{M}$$

for some sufficiently small Δ. To show this, it suffices to prove that $g_j(z_j, s)$ and $g_k(z_k, s)$ coincide on $(V_j \cap V_k) \times \Delta$ for some small Δ. But by our construction of $g_j(z_j, s)$ and $h(s)$, we have

$$g_j(f_{jk}(z_k, s), s) = g_{jk}(g_k(z_k, s), h(s))$$

on $(V_j \cap V_k) \times \Delta$ provided that Δ is sufficiently small. This implies that if $z_j = f_{jk}(z_k, s)$, then

$$g_j(z_j, s) = g_{jk}(g_k(z_k, s), h(s)).$$

As (ζ_j, t) and (ζ_k, t) represent the same point of $\mathcal{M}$ if and only if $\zeta_j = g_{jk}(\zeta_k, t)$, we conclude that $g_j(z_j, s)$ and $g_k(z_k, s)$ coincide on $(V_j \cap V_k) \times \Delta$ for some small Δ. Thus we get a holomorphic map g from $\mathcal{N}_\Delta$ into $\mathcal{M}$ with $g = G_j$ on $V_j \times \Delta$. Moreover, $\hat{\omega} \circ g = h \circ \pi$ follows immediately from (4.43). Therefore, Theorem of Completeness 4.1.1 is proved.

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